Abstract

In topological dynamics, the *Symbolic Extension Entropy Theorem* (SEET) (Boyle and Downarowicz, 2004) describes the possibility of a lossless digitalization of a dynamical system by extending it to a subshift on finitely many symbols. The theorem gives a precise estimate on the entropy of such a symbolic extension (and hence on the necessary number of symbols). Unlike in the measure-theoretic case, where Kolmogorov–Sinai entropy serves as an estimate in an analogous problem, in the topological setup the task reaches beyond the classical theory of measure-theoretic and topological entropy. Necessary are tools from an extended theory of entropy, the *theory of entropy structures* developed in Downarowicz (2005). The main goal of this paper is to prove the analog of the SEET for actions of (discrete infinite) countable amenable groups:

Let a countable amenable group G act by homeomorphisms on a compact metric space X and let $\mathcal{M}_G(X)$ denote the simplex of all G-invariant Borel probability measures on X. A function E_{A} on $\mathcal{M}_G(X)$ equals the extension entropy function h^π of a symbolic extension $\pi : (Y,G) \to (X,G)$, where $h^\pi(\mu) = \sup\{h_\nu(Y,G) : \nu \in \pi^{-1}(\mu)\}$ $(\mu \in \mathcal{M}_G(X))$, if and only if E_{A} is a finite affine superenvelope of the entropy structure of (X,G).

Of course, the statement is preceded by the presentation of the concepts of an entropy structure and its superenvelopes, adapted from the case of $\mathbb{Z}$-actions. In full generality we are able to prove a slightly weaker version of SEET, in which symbolic extensions are replaced by *quasi-symbolic extensions*, i.e., extensions in form of a joining of a subshift with a zero-entropy tiling system. The notion of a tiling system is a subject of several earlier works (e.g. Downarowicz and Huczek (2018) and Downarowicz, Huczek, and Zhang (2019)) and in this paper we review and complement the theory developed there. The full version of the SEET (with genuine symbolic extensions) is proved for groups which are either residually finite or enjoy the so-called *comparison property*. In order to describe the range of

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our theorem more clearly, we devote a large portion of the paper to studying the comparison property. Our most important result in this aspect is showing that all subexponential groups have the comparison property (and thus satisfy the SEET). To summarize, the heart of the paper is the presentation of the following four major topics and the interplay between them:

- Symbolic extensions,
- Entropy structures,
- Tiling systems (and their encodability),
- The comparison property.