Abstract

We describe higher dimensional generalizations of Ramanujan's classical differential relations satisfied by the Eisenstein series E_2, E_4, E_6. Such "higher Ramanujan equations" are given geometrically in terms of vector fields living on certain moduli stacks classifying abelian schemes equipped with suitable frames of their first de Rham cohomology. These vector fields are canonically constructed by means of the Gauss-Manin connection and the Kodaira-Spencer isomorphism. Using Mumford's theory of degenerating families of abelian varieties, we construct remarkable solutions of these differential equations generalizing (E_2, E_4, E_6), which are also shown to be defined over $\mathbf{Z}$.

This geometric framework taking account of integrality issues is mainly motivated by questions in Transcendental Number Theory regarding an extension of Nesterenko's celebrated theorem on the algebraic independence of values of Eisenstein series. In this direction, we discuss the precise relation between periods of abelian varieties and the values of the above referred solutions of the higher Ramanujan equations, thereby linking the study of such differential equations to Grothendieck's Period Conjecture. Working in the complex analytic category, we prove "functional" transcendence results, such as the Zariski-density of every leaf of the holomorphic foliation induced by the higher Ramanujan equations.