Contents

Introduction ix
Motivation ix
Higher Ramanujan equations over $\mathbf{Z}$; Siegel case xi
Interlude: Grothendieck's Period Conjecture xiii
Analytic higher Ramanujan equations, periods, and transcendence xvi
The Hilbert-Blumenthal case and an algebraic independence conjecture xviii
Scholia xix

Terminology and conventions xxi

Acknowledgments xxv

Part 1. The arithmetic theory of the higher Ramanujan equations 1

Chapter 1. Symplectic vector bundles over schemes 3
1.1. Symplectic vector bundles 3
1.2. Lagrangian subbundles 3
1.3. Symplectic bases 4

Chapter 2. Symplectic-Hodge bases of principally polarized abelian schemes 7
2.1. De Rham cohomology of abelian schemes 7
2.2. Symplectic form associated to a principal polarization 7
2.3. Symplectic-Hodge bases of $H^1_{\mathrm{dR}}(X/U)$ 9

Chapter 3. Abelian schemes with real multiplication 11
3.1. Symplectic vector bundles with real multiplication 11
3.2. Principally polarized abelian schemes with real multiplication 12
3.3. Symplectic-Hodge bases 13

Chapter 4. The moduli stacks $\mathcal{B}_g$ and $\mathcal{B}_F$ 15
4.1. The moduli stacks $\mathcal{A}_g$ and $\mathcal{A}_F$ 15
4.2. Definition of the moduli stacks $\mathcal{B}_g$ and $\mathcal{B}_F$ 16
4.3. Siegel parabolic subgroup and proof of Theorem 4.2.2 for $\mathcal{B}_g$ 17
4.4. Proof of Theorem 4.2.2 for $\mathcal{B}_F$ 19

Chapter 5. The tangent bundles of $\mathcal{B}_g$ and $\mathcal{B}_F$; higher Ramanujan vector fields 21
5.1. Horizontal subbundles and linear connections 21
5.2. The Ramanujan subbundle $\mathcal{R}_g \subset T_{\mathcal{B}_g/\mathbf{Z}}$ 22
5.3. The Ramanujan subbundle $\mathcal{R}_F \subset T_{\mathcal{B}_F/\mathbf{Z}}$ 25
5.4. Recollections on the Kodaira-Spencer morphism 26

5.5. The Kodaira-Spencer isomorphism for $\mathcal{A}_g$ and $\mathcal{A}_F$ 28
5.6. The higher Ramanujan vector fields on $\mathcal{B}_g$ 29
5.7. The higher Ramanujan vector fields on $\mathcal{B}_F$ 31

Chapter 6. Integral solution of the higher Ramanujan equations 33
6.1. Higher Ramanujan equations over $\mathcal{B}_g$ 33
6.2. Integral solution of the higher Ramanujan equations; Siegel case 34
6.3. Higher Ramanujan equations over $\mathcal{B}_F$ 35
6.4. Integral solution of the higher Ramanujan equations; Hilbert-Blumenthal case 36

Chapter 7. Representability of $\mathcal{B}_g$ and $\mathcal{B}_F$ by a scheme 39
7.1. Representability by an algebraic space 39
7.2. Representability of $\mathcal{B}_{g,\mathbf{Z}[1/2]}$ by a quasi-projective scheme B_g 41
7.3. B_g is quasi-affine over $\mathbf{Z}[1/2]$ 43

Chapter 8. The case of elliptic curves: explicit equations 45
8.1. Explicit equation for the universal elliptic curve X_1 over B_1 and its universal symplectic-Hodge basis 45
8.2. Explicit formulas for the Ramanujan vector field 47
8.3. Explicit formulas for $\hat{\varphi}_1$ 48

Part 2. The analytic higher Ramanujan equations and periods of abelian varieties 51

Chapter 9. Analytic families of complex tori, abelian varieties, and their uniformization 53
9.1. Relative complex tori 53
9.2. Riemann forms and principally polarized complex tori 54
9.3. The category $\mathcal{A}_g^{\mathrm{an}}$ of principally polarized complex tori of relative dimension g 55
9.4. De Rham cohomology of complex tori 56
9.5. Relative uniformization of complex abelian schemes 58
9.6. Principally polarized complex tori with real multiplication 60

Chapter 10. Analytic moduli spaces of complex abelian varieties with a symplectic-Hodge basis 63
10.1. Descent of principally polarized complex tori 63
10.2. Integral symplectic bases over principally polarized complex tori 64
10.3. Principal (symplectic) level structures 65
10.4. Symplectic-Hodge bases over complex tori 67
10.5. The Hilbert-Blumenthal case 69

Chapter 11. The analytic higher Ramanujan equations 73
11.1. Definition of φ_g and statement of our main theorem in the Siegel case 73
11.2. Preliminary results 74
11.3. Proof of Theorem 11.1.2 76
11.4. Compatibility of φ_g with $\hat{\varphi}_g$ 77
11.5. Analytic higher Ramanujan equations over $\mathcal{B}_F$ 78
11.6. Compatibility of φ_F and $\hat{\varphi}_F$ 81

Chapter 12. Values of φ_g and φ_F; periods of abelian varieties 83
12.1. Fields of periods of abelian varieties and statement of our main theorems 83
12.2. Period matrices 84
12.3. Auxiliary lemmas 85
12.4. Proof of Theorem 12.1.3 86
12.5. Periods of abelian varieties with real multiplication 87

Chapter 13. An algebraic independence conjecture on the values of φ_F 91
13.1. Hirzebruch-Zagier divisors and statement of the conjecture 91
13.2. Periods in the presence of complex multiplication 92
13.3. Grothendieck's Period Conjecture for abelian surfaces with real multiplication 94

Chapter 14. Group-theoretic description of the higher Ramanujan vector fields 97
14.1. Realization of $B_g(\mathbf{C})$ as an open submanifold of $\mathrm{Sp}_{2g}(\mathbf{Z})\backslash \mathrm{Sp}_{2g}(\mathbf{C})$ 97
14.2. Explicit analytic description of the higher Ramanujan vector fields v_{ij} and of φ_g 100
14.3. Group-theoretic description of $\mathcal{B}_F$, v_F, and φ_F 102

Chapter 15. Zariski-density of leaves of the higher Ramanujan foliation 107
15.1. Characterization of the leaves of the higher Ramanujan foliation 107
15.2. Auxiliary results 110
15.3. Statement and proof of our Zariski-density results 112
15.4. Derivatives of modular functions and B_g 113

Appendix A. Gauss-Manin connection on some families of elliptic curves 115
A.1. The Weierstrass elliptic curve 115
A.2. The elliptic curve $X_{/B}$ over $\mathbf{Z}[1/6]$ 116
A.3. The universal elliptic curve X_{1/B_1} over $\mathbf{Z}[1/2]$ 117

Bibliography 119

Index of notation 123