Contents

Abstract

For a fairly general family of L-functions, we survey the known consequences of the existence of asymptotic formulas with power-saving error term for the (twisted) first and second moments of the central values in the family.

We then consider in detail the important special case of the family of twists of a fixed cusp form by primitive Dirichlet characters modulo a prime q, and prove that it satisfies such formulas. We derive arithmetic consequences:

- a positive proportion of central values $L(f \otimes \chi, 1/2)$ are non-zero, and indeed bounded from below;
- there exist many characters χ for which the central L-value is very large;
- the probability of a large analytic rank decays exponentially fast.

We finally show how the second moment estimate establishes a special case of a conjecture of Mazur and Rubin concerning the distribution of modular symbols.

Received by the editor April 7, 2018, and, in revised form, December 5, 2019.

Article electronically published on January 3, 2023.

DOI: https://doi.org/10.1090/memo/1394

2020 *Mathematics Subject Classification.* Primary: 11M06, 11F11, 11F12, 11F66, 11F67, 11L05, 11L40, 11F72, 11T23.

Key words and phrases. L-functions, modular forms, special values of L-functions, moments, mollification, analytic rank, shifted convolution sums, root number, Kloosterman sums, resonator method.

©2023 American Mathematical Society