Abstract

The finite subgroups of $GL_4(\mathbb{Z})$ are classified up to conjugation in Brown, Büllow, Neubüser, Wondratscheck, and Zassenhaus (1978); in particular, there exist 710 non-conjugate finite groups in $GL_4(\mathbb{Z})$. Each finite group G of $GL_4(\mathbb{Z})$ acts naturally on $\mathbb{Z}^{\oplus 4}$; thus we get a faithful G-lattice M with $\mathrm{rank}_{\mathbb{Z}} M = 4$. In this way, there are exactly 710 such lattices. Given a G-lattice M with $\mathrm{rank}_{\mathbb{Z}} M = 4$, the group G acts on the rational function field $\mathbb{C}(M) := \mathbb{C}(x_1, x_2, x_3, x_4)$ by multiplicative actions, i.e. purely monomial automorphisms over $\mathbb{C}$. We are concerned with the rationality problem of the fixed field $\mathbb{C}(M)^G$. A tool of our investigation is the unramified Brauer group of the field $\mathbb{C}(M)^G$ over $\mathbb{C}$. It is known that, if the unramified Brauer group, denoted by $\mathrm{Br}_u(\mathbb{C}(M)^G)$, is non-trivial, then the fixed field $\mathbb{C}(M)^G$ is not rational (= purely transcendental) over $\mathbb{C}$. A formula of the unramified Brauer group $\mathrm{Br}_u(\mathbb{C}(M)^G)$ for the multiplicative invariant field was found by Saltman in 1990. However, to calculate $\mathrm{Br}_u(\mathbb{C}(M)^G)$ for a specific multiplicatively invariant field requires additional efforts, even when the lattice M is of rank equal to 4. There is a direct decomposition $\mathrm{Br}_u(\mathbb{C}(M)^G) = B_0(G) \oplus H_u^2(G, M)$ where $H_u^2(G, M)$ is some subgroup of $H^2(G, M)$. The first summand $B_0(G)$, which is related to the faithful linear representations of G, has been investigated by many authors. But the second summand $H_u^2(G, M)$ doesn't receive much attention except when the rank is ≤ 3. Theorem 1. Among the 710 finite groups G, let M be the associated faithful G-lattice with $\mathrm{rank}_{\mathbb{Z}} M = 4$, there exist precisely 5 lattices M with $\mathrm{Br}_u(\mathbb{C}(M)^G) \neq 0$. In these situations, $B_0(G) = 0$ and thus $\mathrm{Br}_u(\mathbb{C}(M)^G) \subset H^2(G, M)$. The 5 groups are isomorphic to D_4, Q_8, QD_8, $SL_2(\mathbb{F}_3)$,

$GL_2(\mathbb{F}_3)$ whose GAP IDs are (4,12,4,12), (4,32,1,2), (4,32,3,2), (4,33,3,1), (4,33,6,1) respectively in Brown, Büllow, Neubüser, Wondratscheck, and Zassenhaus (1978) and in The GAP Group (2008). Theorem 2. There exist 6079 (resp. 85308) finite subgroups G in $GL_5(\mathbb{Z})$ (resp. $GL_6(\mathbb{Z})$). Let M be the lattice with rank 5 (resp. 6) associated to each group G. Among these lattices precisely 46 (resp. 1073) of them satisfy the condition $\mathrm{Br}_u(\mathbb{C}(M)^G) \neq 0$. The GAP IDs (actually the CARAT IDs) of the corresponding groups G may be determined explicitly. Motivated by these results, we construct G-lattices M of rank $2n+2$, $4n$, $p(p-1)$ (n is any positive integer and p is any odd prime number) satisfying that $B_0(G) = 0$ and $H_u^2(G, M) \neq 0$; and therefore $\mathbb{C}(M)^G$ are not rational over $\mathbb{C}$. For these G-lattices M, we prove that the flabby class $[M]^{fl}$ of M is not invertible. We also construct an example of $(C_2)^3$-lattice (resp. A_6-lattice) M of rank 7 (resp. 9) with $\mathrm{Br}_u(\mathbb{C}(M)^G) \neq 0$. As a consequence, we give a counter-example to Noether's problem for $N \rtimes A_6$ over $\mathbb{C}$ where N is some abelian group.