Abstract

A closed subscheme $X \subset \mathbb{P}^n$ is said to be *determinantal* if its homogeneous saturated ideal can be generated by the $s \times s$ minors of a homogeneous $p \times q$ matrix satisfying $(p-s+1)(q-s+1) = n - \dim X$ and it is said to be *standard determinantal* if, in addition, $s = \min(p,q)$. Given integers $a_1 \leq a_2 \leq \cdots \leq a_{t+c-1}$ and $b_1 \leq b_2 \leq \cdots \leq b_t$ we consider $t \times (t+c-1)$ matrices $\mathcal{A} = (f_{ij})$ with entries homogeneous forms of degree $a_j - b_i$ and we denote by $\overline{W(\underline{b};\underline{a};r)}$ the closure of the locus $W(\underline{b};\underline{a};r) \subset \operatorname{Hilb}^{p(t)}(\mathbb{P}^n)$ of determinantal schemes defined by the vanishing of the $(t-r+1)\times(t-r+1)$ minors of such $\mathcal{A}$ for $\max\{1, 2-c\} \leq r < t$. $W(\underline{b};\underline{a};r)$ is an irreducible algebraic set.

First of all, we compute an upper r-independent bound for the dimension of $W(\underline{b};\underline{a};r)$ in terms of a_j and b_i which is sharp for $r = 1$. In the linear case $(a_j = 1, b_i = 0)$ and cases sufficiently close, we conjecture and to a certain degree prove that this bound is achieved for all r. Then, we study to what extent the family $W(\underline{b};\underline{a};r)$ fills in a generically smooth open subset of the corresponding component of the Hilbert scheme $\operatorname{Hilb}^{p(t)}(\mathbb{P}^n)$ of closed subschemes of $\mathbb{P}^n$ with Hilbert polynomial $p(t) \in \mathbb{Q}[t]$. Under some weak numerical assumptions on the integers a_j and b_i (or under some depth conditions) we conjecture and often prove that $\overline{W(\underline{b};\underline{a};r)}$ is a generically smooth component. Moreover, we also study the depth of the normal module of the homogeneous coordinate ring of $(X) \in W(\underline{b};\underline{a};r)$ and of a closely related module. We conjecture, and in some cases prove, that their codepth is often 1 (resp. r). These results extend previous results on *standard determinantal* schemes to *determinantal* schemes; i.e. previous results of the authors on $W(\underline{b};\underline{a};1)$ to $W(\underline{b};\underline{a};r)$ with $1 \leq r < t$ and $c \geq 2 - r$. Finally, deformations of exterior powers of the cokernel of the map determined by $\mathcal{A}$ are studied and proven to be given as deformations of $X \subset \mathbb{P}^n$ if $\dim X \geq 3$.

The work contains many examples which illustrate the results obtained and a considerable number of open problems; some of them are collected as conjectures in the final section.