Abstract

In the first part we construct algorithms (over $\mathbb{Q}$) which we apply to solve S-unit, Mordell, cubic Thue, cubic Thue–Mahler and generalized Ramanujan–Nagell equations. As a byproduct we obtain alternative practical approaches for various classical Diophantine problems, including the fundamental problem of finding all elliptic curves over $\mathbb{Q}$ with good reduction outside a given finite set of rational primes. The first type of our algorithms uses modular symbols, and the second type combines explicit height bounds with efficient sieves. In particular we construct a refined sieve for S-unit equations which combines Diophantine approximation techniques of de Weger with new geometric ideas. To illustrate the utility of our algorithms we determined the solutions of large classes of equations, containing many examples of interest which are out of reach for the known methods. In addition we used the resulting data to motivate various conjectures and questions, including Baker's explicit *abc*-conjecture and a new conjecture on S-integral points of any hyperbolic genus one curve over $\mathbb{Q}$.

In the second part we establish new results for certain old Diophantine problems (e.g. the difference of squares and cubes) related to Mordell equations, and we prove explicit height bounds for cubic Thue, cubic Thue–Mahler and generalized Ramanujan–Nagell equations. As a byproduct, we obtain here an alternative proof of classical theorems of Baker, Coates and Vinogradov–Sprindžuk. In fact we get refined versions of their theorems, which improve the actual best results in many fundamental cases. We also conduct some effort to work out optimized height bounds for S-unit and Mordell equations which are used in our algorithms of the first part. Our results and algorithms all ultimately rely on the method of Faltings (Arakelov, Paršin, Szpiro) combined with the Shimura–Taniyama conjecture, and they all do not use lower bounds for linear forms in (elliptic) logarithms.

In the third part we solve the problem of constructing an efficient sieve for the S-integral points of bounded height on any elliptic curve E over $\mathbb{Q}$ with given Mordell–Weil basis of $E(\mathbb{Q})$. Here we combine a geometric interpretation of the known elliptic logarithm reduction (initiated by Zagier) with several conceptually new ideas. The resulting "elliptic logarithm sieve" is crucial for some of our algorithms of the first part. Moreover, it considerably extends the class of elliptic Diophantine equations which can be solved in practice: To demonstrate this we solved many notoriously difficult equations by combining our sieve with known height bounds based on the theory of logarithmic forms.