Abstract

Many smoothness spaces in harmonic analysis are decomposition spaces. In this paper we ask: Given two such spaces, is there an embedding between the two?

A decomposition space $\mathcal{D}(\mathcal{Q}, L^p, Y)$ is determined by a covering $\mathcal{Q} = (Q_i)_{i\in I}$ of the frequency domain, an integrability exponent p, and a sequence space $Y \subset \mathbb{C}^I$. Given these ingredients, the decomposition space norm of a distribution g is defined as $\|g\|_{\mathcal{D}(\mathcal{Q},L^p,Y)} = \left\| \left(\left\| \mathcal{F}^{-1}\left(\varphi_i \cdot \widehat{g}\right)\right\|_{L^p}\right)_{i\in I}\right\|_Y$, where $(\varphi_i)_{i\in I}$ is a suitable partition of unity for $\mathcal{Q}$.

We establish readily verifiable criteria which ensure the existence of a continuous inclusion ("an embedding") $\mathcal{D}(\mathcal{Q}, L^{p_1}, Y) \hookrightarrow \mathcal{D}(\mathcal{P}, L^{p_2}, Z)$, mostly concentrating on the case where $Y = \ell_w^{q_1}(I)$ and $Z = \ell_v^{q_2}(J)$. Under suitable assumptions on $\mathcal{Q}, \mathcal{P}$, we will see that the relevant sufficient conditions are $p_1 \leq p_2$ and finiteness of a nested norm of the form

$$\left\| \left(\left\| (\alpha_i \beta_j \cdot v_j / w_i)_{i\in I_j} \right\|_{\ell^t} \right)_{j\in J} \right\|_{\ell^s}, \quad \text{with} \quad I_j = \{ i \in I \,:\, Q_i \cap P_j \neq \varnothing \} \quad \text{for } j \in J \, .$$

Like the sets I_j, the exponents t, s and the weights α, β only depend on the quantities used to define the decomposition spaces.

In a nutshell, in order to apply the embedding results presented in this article, no knowledge of Fourier analysis is required; instead, one only has to study the geometric properties of the involved coverings, so that one can decide the finiteness of certain sequence space norms defined in terms of the coverings.

These sufficient criteria are quite sharp: For almost arbitrary coverings and certain ranges of p_1, p_2, our criteria yield a *complete characterization* for the existence of the embedding. The same holds for arbitrary values of p_1, p_2 under more strict assumptions on the coverings.

We also prove a *rigidity result*, namely that—for $(p_1, q_1) \neq (2, 2)$—two decomposition spaces $\mathcal{D}(\mathcal{Q}, L^{p_1}, \ell_w^{q_1})$ and $\mathcal{D}(\mathcal{P}, L^{p_2}, \ell_v^{q_2})$ can only *coincide* if their "ingredients" are equivalent, that is, if $p_1 = p_2$ and $q_1 = q_2$ and if the coverings $\mathcal{Q}, \mathcal{P}$ and the weights w, v are equivalent in a suitable sense.

The resulting embedding theory is illustrated by applications to α-modulation and Besov spaces. All known embedding results for these spaces are special cases of our approach; often, we improve considerably upon the state of the art.