Abstract

In this paper, we use homotopical algebra (or abstract homotopical methods) to study smooth homotopical problems of infinite-dimensional C^∞-manifolds in convenient calculus. More precisely, we discuss the smoothing of maps, sections, principal bundles, and gauge transformations.

We first introduce the notion of hereditary C^∞-paracompactness along with the semiclassicality condition on a C^∞-manifold, which enables us to use local convexity in local arguments. Then, we prove that for C^∞-manifolds M and N, the smooth singular complex of the diffeological space $C^\infty(M, N)$ is weakly equivalent to the ordinary singular complex of the topological space $\mathcal{C}^0(M, N)$ under the hereditary C^∞-paracompactness and semiclassicality conditions on M. We next generalize this result to sections of fiber bundles over a C^∞-manifold M under the same conditions on M. Further, we establish the Dwyer-Kan equivalence between the simplicial groupoid of smooth principal G-bundles over M and that of continuous principal G-bundles over M for a Lie group G and a C^∞-manifold M under the same conditions on M, encoding the smoothing results for principal bundles and gauge transformations.

For the proofs, we fully faithfully embed the category C^∞ of C^∞-manifolds into the category $\mathcal{D}$ of diffeological spaces and develop the smooth homotopy theory of diffeological spaces via a homotopical algebraic study of the model category $\mathcal{D}$ and the model category $\mathcal{C}^0$ of arc-generated spaces, also known as Δ-generated spaces. Then, the hereditary C^∞-paracompactness and semiclassicality conditions on M imply that M has the smooth homotopy type of a cofibrant object in $\mathcal{D}$. This result can be regarded as a smooth refinement of the results of Milnor, Palais, and Heisey, which give sufficient conditions under which an infinite-dimensional topological manifold has the homotopy type of a CW-complex. We also show that most of the important C^∞-manifolds introduced and studied by Kriegl, Michor, and their coauthors are hereditarily C^∞-paracompact and semiclassical, and hence, results can be applied to them.