

# Contents

Chapter 1. Introduction	1
1.1. Fundamental problems on $C^\infty$ -manifolds	1
1.2. Main results on $C^\infty$ -manifolds	4
1.3. Smooth homotopy theory of diffeological spaces	5
1.4. Notation and terminology	8
1.5. Organization of the paper	9
Chapter 2. Diffeological spaces, arc-generated spaces, and $C^\infty$ -manifolds	11
2.1. Categories $\mathcal{D}$ and $\mathcal{C}^0$	11
2.2. Fully faithful embedding of $C^\infty$ into $\mathcal{D}$	13
2.3. Standard $p$ -simplices and model structure on $\mathcal{D}$	15
2.4. Quillen pairs $(\mathcal{D}, S^{\mathcal{D}})$ and $(\mathcal{C}, R)$	18
Chapter 3. Quillen equivalences between $\mathcal{S}$ , $\mathcal{D}$ , and $\mathcal{C}^0$	21
3.1. Singular homology of a diffeological space	21
3.2. Proof of Theorem 1.5	25
3.3. Proof of Corollary 1.6	28
Chapter 4. Smoothing of continuous maps	31
4.1. Enrichment of cartesian closed categories	31
4.2. Simplicial categories $\mathcal{C}^0$ and $\mathcal{D}$	32
4.3. Function complexes and homotopy function complexes for $\mathcal{C}^0$ and $\mathcal{D}$	33
4.4. Proof of Theorem 1.7	40
Chapter 5. Smoothing of continuous principal bundles	41
5.1. $\mathcal{C}$ -partitions of unity	41
5.2. Principal bundles in $\mathcal{C}$	42
5.3. Fiber bundles in $\mathcal{C}$	44
5.4. Smoothing of principal bundles	46
Chapter 6. Smoothing of continuous sections	49
6.1. Quillen equivalences between the overcategories of $\mathcal{S}$ , $\mathcal{D}$ , and $\mathcal{C}^0$	49
6.2. Enrichment of overcategories	50
6.3. Simplicial categories $\mathcal{C}^0/X$ and $\mathcal{D}/X$	53
6.4. Function complexes and homotopy function complexes for $\mathcal{C}^0/X$ and $\mathcal{D}/X$	54
6.5. Proof of Theorem 1.8	57
Chapter 7. Dwyer-Kan equivalence between $(PDG/X)_{\text{num}}$ and $(PC^0\tilde{G}/\tilde{X})_{\text{num}}$	59
7.1. Enrichment of categories embedded into $\mathcal{M}$	59
7.2. Enriched groupoid $PMG/X$	60

7.3. Smoothing of gauge transformations	61
7.4. Proof of Theorem 1.9	63
Chapter 8. Diffeological polyhedra	65
8.1. Basic properties of two kinds of diffeological polyhedra	65
8.2. $\mathcal{D}$ -homotopy equivalence between two kinds of diffeological polyhedra	68
Chapter 9. Homotopy cofibrancy theorem	75
9.1. Diffeological spaces associated to a covering	75
9.2. Hurewicz cofibrations in $\mathcal{D}$	81
9.3. Proof of Theorem 1.10	87
Chapter 10. Locally contractible diffeological spaces	91
Chapter 11. Applications to $C^\infty$ -manifolds	95
11.1. Proofs of Theorems 1.1-1.3	95
11.2. Classical atlases	96
11.3. Hereditary $C^\infty$ -paracompactness	97
11.4. Hereditarily $C^\infty$ -paracompact, semiclassical $C^\infty$ -manifolds	101
Appendix A. Pathological diffeological spaces	113
Appendix B. Keller's $C_c^\infty$ -theory and diffeological spaces	119
Appendix C. Smooth regularity and smooth paracompactness	123
Bibliography	127