

# Contents

|  |     |
|--|-----|
| Chapter 1. Introduction  | 1   |
| Chapter 2. Corollaries, open problems, and plan of the paper   | 7   |
| 2.1. Indecomposable components of $T$                          | 7   |
| 2.2. Almgren's question and proof of Theorem 1.9               | 9   |
| 2.3. Proof of Theorem 1.8                                      | 9   |
| 2.4. Plan of the proof of Theorem 1.6                          | 13  |
| 2.5. Open problems   | 16  |
| Chapter 3. Stratification and reduction to collapsed points    | 19  |
| 3.1. First variation and monotonicity formula                  | 19  |
| 3.2. Stratification  | 22  |
| 3.3. Proof of Theorem 3.8                                      | 23  |
| 3.4. Proofs of Theorem 3.2 and Corollaries 3.11 and 3.12       | 24  |
| Chapter 4. Regularity for $(Q - \frac{1}{2})$ Dir-minimizers   | 27  |
| 4.1. Preliminaries and proof of Theorem 4.2                    | 29  |
| 4.2. The main frequency function estimate                      | 37  |
| 4.3. Further consequences of the frequency estimate            | 45  |
| 4.4. Blowup: Proof of Theorem 4.5 with $\varphi \equiv 0$      | 47  |
| 4.5. Proof of Theorem 4.5: General case                        | 50  |
| Chapter 5. First Lipschitz approximation and harmonic blow-up  | 53  |
| 5.1. Proof of Theorem 5.5                                      | 56  |
| 5.2. Lipschitz approximation of Sobolev maps                   | 62  |
| 5.3. Proof of Theorem 5.6                                      | 65  |
| Chapter 6. Decay of the excess and uniqueness of tangent cones | 75  |
| 6.1. Hardt-Simon height bound                                  | 77  |
| 6.2. Excess decay  | 80  |
| 6.3. Proof of Theorem 6.3                                      | 87  |
| 6.4. Proof of Corollary 6.4                                    | 90  |
| Chapter 7. Second Lipschitz approximation                      | 91  |
| 7.1. Preliminary observations                                  | 92  |
| 7.2. Proof of Theorem 7.4                                      | 94  |
| Chapter 8. Center manifolds                                    | 99  |
| 8.1. Construction of the center manifolds                      | 99  |
| 8.2. The approximation on the normal bundle of $\mathcal{M}$   | 106 |
| 8.3. Estimates on tilting and optimal planes                   | 109 |

|             |  |     |
|-------------|--|-----|
| 8.4.        | Interpolating functions and linearized system          | 115 |
| 8.5.        | Tilted $L^1$ estimate                                  | 122 |
| 8.6.        | Construction estimates and proof of Theorem 8.13       | 125 |
| 8.7.        | Proof of Cor. 8.17 and 8.21, Prop. 8.20 and Theo. 8.19 | 128 |
| 8.8.        | Proof of Proposition 8.23                              | 129 |
| Chapter 9.  | Monotonicity of the frequency function                 | 133 |
| 9.1.        | Frequency function and main monotonicity formula       | 133 |
| 9.2.        | Poincaré inequality                                    | 135 |
| 9.3.        | Differentiating $H$ and $D$                            | 136 |
| 9.4.        | First variations                                       | 137 |
| 9.5.        | Key identities   | 141 |
| 9.6.        | Estimates on the error terms                           | 142 |
| 9.7.        | Proof of Theorem 9.3                                   | 151 |
| Chapter 10. | Final blow-up argument                                 | 153 |
| 10.1.       | Asymptotics for $\mathcal{D}(r)$                       | 154 |
| 10.2.       | Vanishing of the average                               | 155 |
| 10.3.       | Minimality and convergence in energy                   | 156 |
|             | Bibliography   | 163 |
|             | Index  | 165 |