Abstract

We define an infinite sequence of generalizations, parametrized by an integer $m \geq 1$, of the Stieltjes–Rogers and Thron–Rogers polynomials; they arise as the power-series expansions of some branched continued fractions, and as the generating polynomials for m-Dyck and m-Schröder paths with height-dependent weights. We prove that all of these sequences of polynomials are coefficientwise Hankel-totally positive, jointly in all the (infinitely many) indeterminates. We then apply this theory to prove the coefficientwise Hankel-total positivity for combinatorially interesting sequences of polynomials. Enumeration of unlabeled ordered trees and forests gives rise to multivariate Fuss–Narayana polynomials and Fuss–Narayana symmetric functions. Enumeration of increasing (labeled) ordered trees and forests gives rise to multivariate Eulerian polynomials and Eulerian symmetric functions, which include the univariate mth-order Eulerian polynomials as specializations. We also find branched continued fractions for ratios of contiguous hypergeometric series ${}_rF_s$ for arbitrary r and s, which generalize Gauss' continued fraction for ratios of contiguous ${}_2F_1$; and for $s = 0$ we prove the coefficientwise Hankel-total positivity. Finally, we extend the branched continued fractions to ratios of contiguous basic hypergeometric series ${}_r\phi_s$.