

# Contents

Chapter 1. Introduction	1
1.1. Counting matrices of given rank	2
1.2. Class numbers of unipotent groups	3
1.3. Zeta functions	4
1.4. Groups, graphs, and hypergraphs	7
1.5. Results I: Strong uniformity	8
1.6. Results II: Weak orders and explicit formulae for hypergraphs	11
1.7. Results III: cographs and their models	13
1.8. A recurring example	14
1.9. Results IV and open problems	16
1.10. Outline	17
1.11. Notation	18
Further notation	19
Acknowledgments	20
Chapter 2. Ask zeta functions and modules over polynomial rings	21
2.1. Module representations	21
2.2. Matrices associated with module representations involving free modules	22
2.3. Reminder: Ask zeta functions	23
2.4. Application: Class counting zeta functions of Baer group schemes	24
2.5. Cokernel integrals	25
2.6. Zeta functions associated with modules over polynomial rings	26
Chapter 3. Modules and module representations from (hyper)graphs	29
3.1. Graphs, multigraphs, and hypergraphs	29
3.2. The incidence representation and module associated with a hypergraph	31
3.3. Two adjacency representations and modules associated with a graph	33
3.4. Graphical groups and group schemes	35
Chapter 4. Modules over toric rings and associated zeta functions	37
4.1. Cones and fans	37
4.2. Affine toric schemes and their rational points over DVRs	38
4.3. Zeta functions associated with modules over toric rings	39
4.4. Combinatorial and torically combinatorial modules	39
Chapter 5. Ask zeta functions of hypergraphs	43
5.1. An explicit formula for the ask zeta function of a hypergraph	43
5.2. Ask zeta functions of disjoint unions of hypergraphs	49
5.3. Ask zeta functions of complete unions of hypergraphs	51

5.4.	Four basic operations on hypergraphs	56
5.5.	Analytic properties of ask zeta functions of hypergraphs	58
Chapter 6.	Uniformity for ask zeta functions of graphs	61
6.1.	Weighted signed multigraphs and their adjacency modules	61
6.2.	Multigraph surgery	62
6.3.	Torically torically combinatorial modules are torically combinatorial	68
6.4.	Proof of Theorem 6.4: “Solitary induction”	69
Chapter 7.	Graph operations and ask zeta functions of cographs	73
7.1.	Background on cographs	73
7.2.	Comparing adjacency and incidence modules	73
7.3.	Informal overview of the proof of Theorem 7.1	75
7.4.	Outgoing orientations of forests	76
7.5.	Scaffolds	77
7.6.	Models	81
7.7.	Proof of Theorem 7.24	85
Chapter 8.	Cographs, hypergraphs, and cographical groups	91
8.1.	Proofs of Theorems E–F	91
8.2.	Disjoint unions of hypergraphs and direct products of cographical groups	92
8.3.	Complete unions of hypergraphs and free class-2-nilpotent products of cographical groups	93
8.4.	Threshold graphs	95
8.5.	Bivariate conjugacy class zeta functions associated with cographical group schemes	99
Chapter 9.	Further examples	103
9.1.	Computer calculations: Zeta	103
9.2.	Graphs on at most four vertices	103
9.3.	Graphs on five vertices	104
9.4.	Paths and cycles on at most nine vertices	104
9.5.	The numerator of $W_{(\mathbb{K}_3 \oplus \mathbb{K}_3) \vee \mathbb{K}_2}^+$	104
Chapter 10.	Open problems	113
10.1.	The algebra of graphs	113
10.2.	Connections with statistics on Weyl groups	114
10.3.	Analytic properties	114
	Bibliography	117