Abstract

In previous work with M.C. Fernandes, we found a Lie algebroid symmetry for the Einstein evolution equations. The present work was motivated by the effort to combine this symmetry with the hamiltonian structure of the equations to explain the coisotropic structure of the constraint subset for the initial value problem. In this paper, we extend the notion of hamiltonian structure from Lie algebra actions to general Lie algebroids over presymplectic manifolds. Application of this construction to the problem in general relativity is still work in progress.

After comparing a number of possible compatibility conditions between an anchor map $A \to TM$ on a vector bundle A and a presymplectic structure on the base M, we choose the most natural of them, best formulated in terms of a suitably chosen connection on A. We define a notion of momentum section of A^*, and, when A is a Lie algebroid, we specify a condition for compatibility with the Lie algebroid bracket. Compatibility conditions on an anchor, a Lie algebroid bracket, a momentum section, a connection, and a presymplectic structure are then the defining properties of a hamiltonian Lie algebroid. For an action Lie algebroid with the trivial connection, the conditions reduce to those for a hamiltonian action. We show that the clean zero locus of the momentum section of a hamiltonian Lie algebroid is a coisotropic submanifold. To define morphisms of hamiltonian Lie algebroids, we express the structure in terms of a bigraded algebra generated by Lie algebroid forms and de Rham forms on its base. We give an Atiyah-Bott type characterization of a bracket-compatible momentum map; it is equivalent to a closed basic extension of the presymplectic form, within the generalization of the BRST model of equivariant cohomology to Lie algebroids. We show how to construct a groupoid by reduction of an action Lie groupoid $G \times M$ by a subgroup H of G which is not necessarily normal, and we find conditions which imply that a hamiltonian structure descends to such a reduced Lie algebroid.

We consider many examples and, in particular, find that the tangent Lie algebroid over a symplectic manifold is hamiltonian with respect to some connection if and only if the symplectic structure has a nowhere vanishing primitive. Recent results of Stratmann and Tang show that this is the case whenever the symplectic structure is exact.