Abstract

We study continuous bounded cohomology of totally disconnected locally compact groups with coefficients in a non-Archimedean valued field $\mathbb{K}$. To capture the features of classical amenability that induce the vanishing of bounded cohomology with real coefficients, we start by introducing the notion of normed $\mathbb{K}$-amenability, of which we prove an algebraic characterization. It implies that normed $\mathbb{K}$-amenable groups are locally elliptic, and it relates an invariant, the norm of a $\mathbb{K}$-amenable group, to the order of its discrete finite p-subquotients, where p is the characteristic of the residue field of $\mathbb{K}$. Moreover, we prove a characterization of discrete normed $\mathbb{K}$-amenable groups in terms of vanishing of bounded cohomology with coefficients in $\mathbb{K}$.

The algebraic characterization shows that normed $\mathbb{K}$-amenability is a very restrictive condition, so the bounded cohomological one suggests that there should be plenty of groups with rich bounded cohomology with trivial $\mathbb{K}$ coefficients. We explore this intuition by studying the injectivity and surjectivity of the comparison map, for which surprisingly general statements are available. Among these, we show that if either $\mathbb{K}$ has positive characteristic or its residue field has characteristic 0, then the comparison map is injective in all degrees. If $\mathbb{K}$ is a finite extension of $\mathbb{Q}_p$, we classify unbounded and non-trivial quasimorphisms of a group and relate them to its subgroup structure. For discrete groups, we show that suitable finiteness conditions imply that the comparison map is an isomorphism; this applies in particular to finitely presented groups in degree 2.

A motivation as to why the comparison map is often an isomorphism, in stark contrast with the real case, is given by moving to topological spaces. We show that over a non-Archimedean field, bounded cohomology is a cohomology theory in the sense of Eilenberg–Steenrod, except for a weaker version of the additivity axiom which is however equivalent for finite disjoint unions. In particular there exists a Mayer–Vietoris sequence, the main missing piece for computing real bounded cohomology.