Preface

This volume, commissioned by the International Mathematical Union, is part of the activities celebrating 2000 as the Year of Mathematics. It is inspired by the famous list of problems that Hilbert proposed 100 years ago, but it has the more general purpose of describing the state of mathematics at the end of the 20th century.

Hilbert's problems have stood the test of time remarkably well. They vary in character, from the comparatively easy to the almost impossible, but collectively they convey a clear impression of mathematics in 1900. A list of such problems can provide a useful focus : some problems open doors, some problems close doors, and some remain curiosities, but all sharpen our wits and act as a challenge and a test of our ingenuity and techniques. As Gowers says in his article, solving problems can be either the road to understanding or the purpose of understanding.

In fact, the influence of Hilbert's problems on 20th century mathematics can be exaggerated. Certainly Hilbert captured the headlines (or at least he would have done if headline capturing had been as advanced an art as it now is), but it was Hilbert's own mathematical work that had much more influence. At the risk of oversimplification, one could say that Hilbert's formal approach dominated mathematics for the first half of the 20th century, with Bourbaki as his most famous disciple. The focus on axiomatics and the in-depth development of specific areas was, for a while, remarkably successful. This was followed by a period of hybridisation, where specialities were put together (e.g., Algebraic Topology or Topological Groups). Eventually, the latter part of the 20th century saw a return to a less constrained view of mathematics, more in the spirit of Poincaré (as expounded by Arnold), with its emphasis on geometrical thinking, even in areas such as algebra or number theory. It is worth noting that Topology does not figure among the list of Hilbert's problems, but Poincaré, in his address to the 1908 Congress, highlighted it (or Analysus Situs as it was then called) as an important area for the future.

It was widely recognised that no single person could now hope to emulate Hilbert by producing a corresponding list of problems. This is not just undue modesty on the part of contemporary mathematicians. It is more a sober reflection on the enormous range and diversity of mathematics in the year 2000. This volume is therefore a collective effort, but the result is not simply a longer list of problems. Different mathematicians have responded in different ways to the challenge. Some have tried to follow the Hilbert paradigm, but restricted themselves to covering a smaller area (e.g., Jones, Smale, Yau). Others write more from a personal or philosophical viewpoint (Kirwan, Manin, Ruelle) or review likely developments in certain areas (Baker-Wusholtz, Donaldson, McDuff, Wiles). Some concentrate on more detailed problems in depth (Connes, Kazhdan, Mazur), and it is noteworthy that Connes reformulates the Riemann Hypothesis (one of Hilbert's problems) with the full machinery of 20th century mathematics.

In addition to many articles on theoretical physics, there are a few on other important areas of applied mathematics (Lax, Lions, Majda), which stress the interplay between computation, numerical analysis, and partial differential equations, while computational complexity is briefly touched on by Smale. Obviously this does not do justice to the full range of applications of mathematics now and in the future. Probability, for example, is not covered; neither is the key link between logic and computing. It would not be a rash prediction to forecast that real-world problems (including those arising from the continuing computer revolution) will have a profound impact on the development of mathematics in the next century. The basis of such a prediction is as much sociological as scientific, for reasons that I shall try to explain.

Mathematics is affected, not only by other sciences, but also by changes in society. The idea that science, including mathematics, is a "social construct" has gained much notoriety and can be pernicious, but it contains a grain of truth. A cursory look at history and geography will show that the kind of science we do, the way we do it, and the speed and scale of scientific advance are certainly affected by the society in which we live. We need freedom to think and exchange ideas without fear of the Inquisition or its modern equivalents. We need education and books to learn about our heritage, we need wealth to support us, and we need a rich intellectual environment if we are to achieve our full potential. In return, we must make our contribution to society - to its culture, its science, and its economy. Tangible benefits will be expected.

The 20th century has seen enormous political and social change (not all beneficial). One outcome is a vast increase in those provided with mathematical opportunity. Technology has provided us with instant world-wide communication, and economic prosperity has enabled thousands of mathematicians to make a livelihood. Perhaps the Newtons, Gausses and Ramanujans have not increased in proportion, but other factors make up for this. The 20th century has transformed mathematics from a cottage industry run by a few semi-amateurs into a world-wide industry run by an army of professionals.

The 21st century will almost certainly transform mathematics as a human activity yet again. When China produces mathematicians on a scale proportional to its population and when electronic communication has reached maturity, the scene will barely be recognisable. For centuries, physics has had a close symbiotic relationship with mathematics, and Witten's forecast is that the 21st century will see this rise to new heights. But what of the future of Biology? Many predict that understanding the brain will be the major challenge of the next century and, while it would be presumptuous of mathematicians to claim that they will solve the problem, it is not unreasonable to think that mathematics may have a useful part to play. There are also emerging mathematical problems in the handling of the vast data bank produced by the Human Genome Project.

As we look at all the changes that have taken place in mathematics during our lifetime, and the greater changes that are to come, one might become pessimistic : can mathematics continue at this ever-increasing rate and still remain the subject we love? I personally remain optimistic and there are two objective reasons for this optimism. The first is the long history and continuity of the subject. If Newton, Gauss, or even Archimedes were to return, I believe that, after a short course to learn the new jargon, they would understand and even approve of the progress that has been made (though Gauss might say that he had some unpublished papers in his drawer ...). The second reason for optimism is that mathematics has shown a consistent ability to renew itself by a synthesis of preceding work and an infusion of new ideas, some of which originate in the real world. Only in this way is it possible for young mathematicians to keep pushing ahead. This process of rejuvenation and evolution is the theme that concludes Manin's contribution.

In addition to listing his famous problems, Hilbert, in his 1900 address, also indulged in philosophical remarks on mathematics, remarks which are still relevant a century later and not dissimilar to the views I have tried to express as a distillation of the articles in this volume. His words speak for themselves, and the final passage of his address is reproduced below (in English translation).

> The problems mentioned are merely samples of problems, yet they will suffice to show how rich, how manifold and how extensive the mathematical science of to-day is, and the question is urged upon us whether mathematics is doomed to the fate of those other sciences that have split up into separate branches, whose representatives scarcely understand one another and whose connection becomes ever more loose. I do not believe this nor wish it. Mathematical science is in my opinion an indivisible whole, an organism whose vitality is conditioned upon the connection of its parts. For with all the variety of mathematical knowledge, we are still clearly conscious of the similarity of the logical devices, the *relationship* of the *ideas* in mathematics as a whole and the numerous analogies in its

> different departments. We also notice that, the farther a mathematical theory is developed, the more harmoniously and uniformly does its construction proceed, and unsuspected relations are disclosed between hitherto separate branches of the science. So it happens that, with the extension of mathematics, its organic character is not lost but only manifests itself the more clearly.
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> But, we ask, with the extension of mathematical knowledge will it not finally become impossible for the single investigator to embrace all departments of this knowledge? In answer let me point out how thoroughly it is ingrained in mathematical science that every real advance goes hand in hand with the invention of sharper tools and simpler methods which at the same time assist in understanding earlier theories and cast aside older more complicated developments. It is therefore possible for the individual investigator, when he makes these sharper tools and simpler methods his own, to find his way more easily in the various branches of mathematics than is possible in any other science.
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> The organic unity of mathematics is inherent in the nature of this science, for mathematics is the foundation of all exact knowledge of natural phenomena. That it may completely fulfil this high mission, may the new century bring it gifted masters and many zealous and enthusiastic disciples.[1]

Another great mathematician, from the generation succeeding Hilbert, was John von Neumann. His interests covered a very wide range of pure and applied mathematics, and he had a deep appreciation of the links between them. Perhaps I can do no better than end by quoting from his article on "The Mathematician".

> I think that it is a relatively good approximation to truth - which is much too complicated to allow anything but approximations - that mathematical ideas originate in empirics, although the genealogy is sometimes long and obscure. But once they are so conceived, the subject begins to live a peculiar life of its own and is better compared to a creative one, governed by almost entirely aesthetical motivations, than to anything else and, in particular, to an empirical science. ... But there is a grave danger that the subject will develop along the line of least resistance, that the stream, so far from its source, will separate into a multitude of insignificant branches, and that the

[1]Reprinted from Hilbert, David, "Mathematical Problems," *Bulletin of the American Mathematical Society* **8** (1902), American Mathematical Society, Providence, RI, pp. 478–479.

discipline will become a disorganised mass of details and complexities. In other words, at a great distance from its empirical source, or after much "abstract" inbreeding, a mathematical subject is in danger of degeneration. At the inception the style is usually classical; when it shows signs of becoming baroque, then the danger signal is up.[2]

Michael Atiyah
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[2]Reprinted with permission from von Neumann, John, "The Mathematician" in *The Works of the Mind*, edited for the Committee on Social Thought by Heywood, Robert B., The University of Chicago Press, 1947.