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Foreword

Financial mathematics is at present going through a period of intensive development, especially in the area connected with contemporary *stochastic analysis*. It is the methods of the general theory of random processes that have turned out to be most suitable for an adequate description of the evolution of *basic* (bonds and stocks) and *derivative* (forwards, futures, options, and so on) securities.

Historically, the first work (1900) in this direction was the dissertation of Bachelier [13], a student of Poincaré who, several years before Einstein and 23 years before Wiener, gave a mathematical definition of the concept of 'Brownian motion', used it to model the dynamics of stock prices, and gave a formula for the investment cost of an option. The main deficiency of Bachelier's model, which was the possible negativity of the stock prices, was removed in 1965 by the well-known economist Samuelson, who proposed a *geometric Brownian motion* for describing these prices. This model now bears the names of Black and Scholes, who in 1973 [15] obtained precise formulas for computing the fair price and hedging strategies for European options in the framework of the model.

Employing the heuristic argument that stock prices are either rising or falling at any moment of time, Cox, Ross, and Rubinstein [19] proposed regarding these changes as *discrete* and introduced a *binomial model* of a financial market. They showed that the binomial model has a geometric Brownian motion 'as a limit', and the formula obtained for a fair price converges to the Black–Scholes formula.

These now-classical papers have become a direct basis for the application and development of methods from contemporary stochastic analysis in the mathematical theory of finance. It is in this direction, with the use of elements of functional analysis and convex analysis, that deep results have been obtained about the structure of prices and about the properties of arbitrage and completeness of a financial market.

The goal of this book is to present, in a sufficiently self-contained form, the methods and results of the contemporary theory of financial computations for a discrete market. It gives a representation of basic techniques in stochastic analysis: martingales, semimartingales, stochastic exponents, Itô's formula, Girsanov's theorem, and so on. The discreteness of the models considered above leads to a whole series of technical simplifications, and often to greater clarity of the results obtained. Yet at the same time, this discrete theory contains in itself many elements of the very complex techniques and problems in the general theory. Therefore, the book can be regarded as a sufficiently broad introduction to the contemporary mathematics of financial computations with derivative securities.

In large part this book is based on the material and approaches expounded in [12], and it represents the content of the course of lectures "Stochastic analysis in finance" given by the author in 1994–1997 in the Mechanics and Mathematics

Department of Moscow State University. This explains its theoretical character and direction.

The author sincerely thanks A. N. Shiryaev, Yu. M. Kabanov, and D. O. Kramkov for many useful discussions and for their help, criticism, and constant support. The book was published at the proposal of the scientific publishing house "TVP". The author is very grateful to V. I. Khokhlov, both for making this proposal and for the amount of work he put into editing the book.

November 22, 1998 *A. V. Mel′nikov*