

Contents

Foreword	xi
Notation	xiii
Lecture 1. Basic concepts and objects of a financial market	1
§1 Financial markets	1
§2 Basic securities	1
§3 Derivative instruments of a financial market	2
§4 Activities of an investor on a financial market	4
§5 Interest rates and discount rates	5
§6 Stock exchanges, clearing houses, and interbank markets	6
§7 Basics of a mathematical model of the dynamics of prices on a financial market	6
Lecture 2. The elements of discrete stochastic analysis	7
§1 A stochastic base	7
§2 Martingales	8
§3 Local martingales and stochastic integrals	12
§4 Semimartingales	13
§5 Change of measure and martingales	13
§6 Stochastic equations and stochastic exponentials	14
§7 Itô's formula	17
§8 Optimal stopping of a stochastic sequence	17
Problems	19
Lecture 3. A stochastic model for a financial market. Arbitrage and completeness	21
§1 A model for a market and investment strategies	21
§2 Martingale measures and arbitrage	22
§3 Martingale measures and completeness	27
Problems	30
Lecture 4. Pricing European options in complete markets. The binomial model and the Cox–Ross–Rubinstein formula	31
§1 Contingent claims and European options	31
§2 General formulas for computing prices and hedging strategies for European options	33
§3 The binomial model of a (B, S) -market. Its properties of no-arbitrage and completeness	35

§4 Options with contingent claims of the form $f = f(S_N)$ in the binomial model	38
§5 The Cox–Ross–Rubinstein formula	40
§6 An example of option pricing on a currency market	41
§7 Pricing and hedging contingent claims attainable with positive probability	42
Problems	44
Lecture 5. Pricing and hedging American options in complete markets	47
§1 Dynamic contingent claims and American options	47
§2 Pricing American options as an optimal stopping problem	48
§3 The methodology of pricing American options	52
Problems	53
Lecture 6. Financial computations on a complete market with the use of nonself-financing strategies	55
§1 G -financing strategies	55
§2 Pricing European options with the use of G -financing strategies	56
§3 Pricing American options	59
Problems	61
Lecture 7. Incomplete markets. Pricing of options and problems of minimizing risk	63
§1 Ask and bid prices. An example of an incomplete market	63
§2 Formulas for computing the ask and bid prices for convex contingent claims	65
§3 On financial computations taking into account the risks of hedging contingent claims	68
Problems	72
Lecture 8. The structure of prices of other instruments of a financial market. Forwards, futures, bonds	73
§1 Forward and futures contracts: The term structure of forward and futures prices	73
§2 Forward markets and the structure of an optimal investment portfolio	76
§3 Bonds: Yield, duration, term structure of prices and interest rates	79
Problems	83
Lecture 9. The problem of optimal investment	85
§1 Statement of the investment problem, utility functions	85
§2 The yield of an investment portfolio and its optimization	86
§3 Optimal investment in the binomial model of a (B, S) -market	89
Problems	92
Lecture 10. The concept of continuous models. Limiting transitions from a discrete market to a continuous one. The Black–Scholes formula	93
§1 (B, S, Δ) -markets and continuous models	93
§2 The Black–Scholes formula	95
Problems	97

CONTENTS

ix

Appendix 1	99
Appendix 2	105
Appendix 3	109
Hints for solving the problems	121
Bibliography	129
Subject index	131

Foreword

Financial mathematics is at present going through a period of intensive development, especially in the area connected with contemporary *stochastic analysis*. It is the methods of the general theory of random processes that have turned out to be most suitable for an adequate description of the evolution of *basic* (bonds and stocks) and *derivative* (forwards, futures, options, and so on) securities.

Historically, the first work (1900) in this direction was the dissertation of Bachelier [13], a student of Poincaré who, several years before Einstein and 23 years before Wiener, gave a mathematical definition of the concept of ‘Brownian motion’, used it to model the dynamics of stock prices, and gave a formula for the investment cost of an option. The main deficiency of Bachelier’s model, which was the possible negativity of the stock prices, was removed in 1965 by the well-known economist Samuelson, who proposed a *geometric Brownian motion* for describing these prices. This model now bears the names of Black and Scholes, who in 1973 [15] obtained precise formulas for computing the fair price and hedging strategies for European options in the framework of the model.

Employing the heuristic argument that stock prices are either rising or falling at any moment of time, Cox, Ross, and Rubinstein [19] proposed regarding these changes as *discrete* and introduced a *binomial model* of a financial market. They showed that the binomial model has a geometric Brownian motion ‘as a limit’, and the formula obtained for a fair price converges to the Black–Scholes formula.

These now-classical papers have become a direct basis for the application and development of methods from contemporary stochastic analysis in the mathematical theory of finance. It is in this direction, with the use of elements of functional analysis and convex analysis, that deep results have been obtained about the structure of prices and about the properties of arbitrage and completeness of a financial market.

The goal of this book is to present, in a sufficiently self-contained form, the methods and results of the contemporary theory of financial computations for a discrete market. It gives a representation of basic techniques in stochastic analysis: martingales, semimartingales, stochastic exponents, Itô’s formula, Girsanov’s theorem, and so on. The discreteness of the models considered above leads to a whole series of technical simplifications, and often to greater clarity of the results obtained. Yet at the same time, this discrete theory contains in itself many elements of the very complex techniques and problems in the general theory. Therefore, the book can be regarded as a sufficiently broad introduction to the contemporary mathematics of financial computations with derivative securities.

In large part this book is based on the material and approaches expounded in [12], and it represents the content of the course of lectures “Stochastic analysis in finance” given by the author in 1994–1997 in the Mechanics and Mathematics

Department of Moscow State University. This explains its theoretical character and direction.

The author sincerely thanks A. N. Shiryayev, Yu. M. Kabanov, and D. O. Kramkov for many useful discussions and for their help, criticism, and constant support. The book was published at the proposal of the scientific publishing house "TVP". The author is very grateful to V. I. Khokhlov, both for making this proposal and for the amount of work he put into editing the book.

November 22, 1998

A. V. Mel'nikov