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Preface

This book was written in 1996, two hundred years after 1796, which was a very fruitful year for the great Gauss, who made many fundamental contributions to modern number theory. Gauss was in his late teens at the time. On March 30 he discovered a method of construction of a regular 17-gon. On April 8 he proved the quadratic reciprocity law (see §2.2 in this volume), which he himself called a gem. On May 31 he conjectured what would later be called "the prime number theorem" concerning the distribution of prime numbers. On July 10 he proved that any natural number can be expressed as a sum of at most three triangular numbers (see §0.5). On October 1 he obtained a result on the number of solutions for an equation with coefficients in a finite field, which had a great impact on mathematics in later eras. All these contributions are discussed in these volumes, *Number Theory 1, 2, 3*.

One, two, three, four... as naive as it is, the world of numbers encompasses many wonders that fascinated young Gauss. A discovery in one epoch induces a more profound discovery by the following generation. A hundred years later, in 1896, the prime number theorem was proved. After some 120 years, the quadratic reciprocity law had grown into the class field theory. After 150 years, André Weil, who had examined Gauss's result of October 1, proposed the so-called Weil conjectures. These conjectures influenced a great deal of algebraic geometry in the twentieth century. The brilliance of the gems polished by Gauss has increased through the efforts of the mathematicians of following generations. It is said that there is no unexplored place on the earth any longer, but the world of numbers is still full of mysteries. That makes us think of the profoundness and richness of nature.

Wandering naively in the wonderland of numbers, we would like to describe in this book the intricate world of numbers that modern number theory has discoverd. We will be very happy if the reader discovers the wonders of numbers and the grandeur of nature.

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Preface to the English Edition

The authors hope that the readers enjoy the wonderful world of modern number theory through the book.

Our special thanks are due to Dr. Masato Kuwata, who not only translated the Japanese edition into English but also suggested many improvements on the text so that the present English edition is more readable than the original Japanese edition.

Objectives and Outline of these Books

In thses books, *Number Theory 1, 2, 3*, we introduce core theories in modern number theory, such as class field theory, Iwasawa theory, the theory of modular forms, etc. The structure of this book is as follows.

The starting point of number theory is astonishment at the wonders of numbers. The work of Fermat, who is considered to be a founding father of modern number theory, illustrates very well the wonder of numbers. We first discuss the work of Fermat on number theory in the introduction to *Number Theory 1*. The reader will learn how mathematicians of later eras little by little found a fascinating world behind each fact discovered by Fermat. In *Number Theory 1* we study some important topics in modern number theory, such as elliptic curves (Chapter 1), p-adic numbers (Chapter 2), the ζ-function (Chapter 3), and number fields (Chapter 4). These chapters are more or less independent; the material in the earlier chapters is not necessary to understand each succeeding chapter. Chapters 2 and 3 may be easier to read than Chapter 1. The reader should not hesitate to skip parts that are difficult to understand.

Number Theory 2 is devoted to class field theory. We also study the ζ-function once again. In *Number Theory 3* we explain Iwasawa theory and the theory of modular forms, before coming back to elliptic curves once again.

These books are part of the series *Fundamentals of Modern Mathematics*, but we were not satisfied with the introduction of fundamentals. We tried to include today's developments in number theory. For example, we included some important theories developed in recent years, such as the arithemetic theory of elliptic curves, which is part of arithmetic algebraic geometry, and Iwasawa theory, to which we did not find an introduction elsewhere. We hope that we convey the best of modern number theory.

We wanted to include more topics, but we had to omit many of them due to the limitation on the number of pages. We regret that we could not mention Diophantine approximations and transcendental number theory, both of which are seeing new developments in recent years.

Prerequisites to *Number Theory 1* are the fundamentals of groups, rings and fields. In *Number Theory 2* we recommend that the reader be familiar with Galois theory.

The reader is advised to write down simple and easy examples on scratch paper. Just as astronomical observations are indispensable to the study of astronomy, it is indispensable to observe the numbers in order to study number theory. The wonders are there to be discovered. Also, number theory has a long history, which teaches us interesting lessons. We advise you to take an interest in the history of mathematics.