Contents

Preface

The book is devoted to the study of optimal control problems for distributed parameter systems, i.e., for systems described by a boundary value problem for partial differential equations. A typical optimal control problem can be written in the form

$$J(y,u) \to \inf, \tag{0.1}$$

$$F(y,u) = 0, \tag{0.2}$$

$$u \in U_\partial, \tag{0.3}$$

where $J(\cdot,\cdot) : Y \times U \to \mathbb{R}$ is a functional, $F(\cdot,\cdot) : Y \times U \to V$ is an operator, Y, U, V are Banach spaces, and $U_\partial \subset U$. The spaces Y and U usually referred to as the space of states and the space of controls. The variables y and u are the state and the control respectively. The functional J is called the cost functional. The set U_∂ is a set of constraints. The operator $F(y,u)$ is given by some boundary value problem for partial differential equations.

The simplest example of the problem (0.1)–(0.3) is the following optimal control problem for a distributed parameter system in a bounded domain $\Omega \subset \mathbb{R}^d$ with boundary $\partial\Omega$ of class C^∞:

$$J(y,u) = \int_\Omega (|y(x) - w(x)|^2 + N|u(x) - f(x)|^2)\, dx \to \inf, \tag{0.4}$$

$$\Delta y(x) = u(x), \quad y\big|_{\partial\Omega} = 0, \tag{0.5}$$

$$u(x) \in U_\partial, \tag{0.6}$$

where $w \in L_2(\Omega)$ and $f \in L_2(\Omega)$ are given functions, $N > 0$ is a parameter (the so-called cost parameter), and U_∂ is a convex closed subset in the space $L_2(\Omega)$. The state $y(x)$ and the control $u(x)$ are unknown. Thus, the abstract equation (0.2) is given by the relations (0.5) which for a fixed $u \in L_2(\Omega)$ can be regarded as the Dirichlet problem for the Laplace operator.

We can begin to study the problem (0.4)–(0.6) by solving the boundary value problem (0.5), i.e., by expressing y in terms of u. Then we substitute the result in (0.4), and reduce the problem under consideration to the minimization problem for a functional depending only on u in U_∂. Such an approach is used for many optimal control problems for distributed parameter systems described by well-posed boundary value problems.

However, this method does not always work. In more complicated situations, another approach is possible, which is based on some properties of the functional J.

Let the functional $J(y,u)$ be bounded from below and tend to $+\infty$ as $(y,u) \to \infty$. For such a functional we can considerably ease the requirements on the boundary value problem (0.2), thus including in the framework of the theory those extremal problems for which the boundary value problem (0.2) is not necessarily well posed in the sense of Hadamard. For example, in this book we study the following optimal control problem for a distributed parameter system in a domain $\Omega \subset \mathbb{R}^d$ whose boundary $\partial\Omega$ consists of two disjoint manifolds Γ_0 and Γ_1:

$$\begin{aligned} J(y, u_1, u_2) = &\int_{\Gamma_0} |y(x) - w(x)|^2 d\sigma \\ &+ \int_{\Gamma_1} (|u_1(x) - f_1(x)|^2 + |u_2(x) - f_2(x)|^2)\, d\sigma \to \inf, \end{aligned} \tag{0.7}$$

$$\Delta y(x) = 0,\ x \in \Omega, \quad y\big|_{\Gamma_1} = u_1, \quad \partial_n y\big|_{\Gamma_1} = u_2, \tag{0.8}$$

$$u_1 \in U_\partial, \quad u_2 \in V_\partial, \tag{0.9}$$

where $w \in L_2(\Gamma_0)$, $f_i \in L_2(\Gamma_1)$, $i = 1, 2$, are given functions, ∂_n is the differentiation along the vector field of outward normals to Γ_1, y, u_1, u_2 are unknown functions, U_∂ and V_∂ are convex closed subsets in the space $L_2(\Gamma_1)$. Typical examples of U_∂ and V_∂ are

$$\begin{aligned} &\{u \in L_2(\Gamma_1) : 0 \leqslant u(x) \leqslant 1 \text{ for almost all } x \in \Gamma_1\}, \\ &\{u \in L_2(\Gamma_1) : u(x) \leqslant 0 \text{ for almost all } x \in \Gamma_1\}. \end{aligned} \tag{0.10}$$

For fixed u_1 and u_2 the relations (0.8) define the Cauchy problem for the Laplace operator, which is a classical example of an ill-posed (in the sense of Hadamard) boundary value problem.

The fact that u_1 and u_2 in the extremal problem are not fixed allows us to analyze successfully the problem (0.7)–(0.9).

One of the main goals of this book is to construct a general theory of optimal control of a distributed parameter system of the form (0.1)–(0.3), where (0.2) can be given by not only a well-posed (in the sense of Hadamard) linear or nonlinear boundary value problem, but also by an ill-posed problem such as the Cauchy problem for the elliptic equation (0.8), or a (well-posed or ill-posed) boundary value problem for the Navier–Stokes equations in the space of smooth vector fields in a two- or three-dimensional domain, and so on.

Chapter 1 deals with the existence of solutions to optimal control problems for distributed systems. First, for an abstract extremal problem of the form (0.1)–(0.3), we formulate sufficient conditions for the existence of a solution. Assuming that these conditions are satisfied, we prove the solvability of the extremal problem (0.1)–(0.3). The nontriviality, coercivity, and compactness conditions are crucial here (the compactness condition appears only if equation (0.2) is nonlinear with respect to y).

The nontriviality condition consists in the requirement that the set of pairs (y, u) satisfying (0.2) and (0.3) is nonempty.

The coercivity condition is similar to the coercivity condition which appears when we use variational methods to solve elliptic boundary value problems.

The compactness condition is similar to the compactness condition in the theory of nonlinear boundary value problems.

In Chapter 1 we mainly apply the above existence theorem to a large class of concrete optimal control problems, including the cases where equation (0.2) is described by an ill-posed (in the sense of Hadamard) boundary value problem. Usually, this reduces to the verification of the nontriviality, coercivity, and compactness conditions. However, in a number of cases, such a verification is too complicated and it turns out to be simpler to prove the existence theorem for the abstract problem directly.

The method of proving the existence theorem presented in Chapter 1 was proposed by the author [**51**]–[**53**] for the case of optimal control problems for the Navier–Stokes equations with positive and negative viscosity. Later, J.-L. Lions [**110**] used a similar method in the study of a large class of singular optimal control problems.

Chapter 2 deals with necessary conditions for an extremum in optimal control problems. The set of relations describing such necessary conditions, together with the constraints (0.2), (0.3) of the initial problem, is called an *optimality system*. To deduce optimality systems, we use various versions of the Lagrange principle. One version convenient for applications to a large class of optimal control problems was proved by Ioffe and Tikhomirov [**89**]. However, in many singular optimal control problems the theorem of Ioffe and Tikhomirov is not suitable.

J.-L. Lions [**110**] proposed a method of deducing optimality systems for a large class of singular problems. This method is based not on the Lagrange principle but on using a penalty function.

In Chapter 2, we justify versions of the Lagrange principle that can be applied not only to singular optimal control problems studied in the book by J.-L. Lions [**110**], but also to a number of problems that were indicated in [**110**] as unsovled problems. The solutions of these problems are presented in Chapter 2.

Chapter 2 is mainly devoted to the deduction of optimality systems for certain extremal problems whose solvability is proved in Chapter 1.

The problems considered in Chapters 1 and 2 can be divided into two groups. The first group contains problems for which the solvability theorems and the deduction of an optimaliy system have been known for a long time. We present them only to demonstrate the natural character and simplicity of methods discussed in this book.

The second group consists of optimal control problems for ill-posed or singular distributed systems. The results concerning the solvability, and especially the deduction of optimality systems for these problems, are new and have not been presented earlier in monographs.

Examples of such problems are the optimization problem for ill-posed boundary value problems (the Cauchy problem for the Laplace operator, the backward heat equation) with sets of constraints U_∂ that have an empty interior in L_2 (as, for example, the set (0.10)).

Among assumptions used in the proof of solvability theorems for optimal control problems, the nontriviality condition plays a special role. It is obvious that this condition is essential for the solvability of the extremal problem because, if it fails, i.e., if the system (0.2), (0.3) has no solution, then the set of pairs (y, u) over which the functional (0.1) has to be minimized is empty. As usual, it is easy to verify the nontriviality condition if equation (0.2) is described by a well-posed boundary value problem, where y is the desired function and u is the data of the problem. However, if (0.2) is given by an ill-posed boundary value problem, the situation changes dramatically and some special methods are required for the verification of the nontriviality condition. This topic is new, so it contains more unsolved problems than definitive results. However, in the case where the mapping $F(y, u)$ is an affine function of y and u, contours of a general theory become visible. The results obtained in this direction are presented in Chapter 3. We prove some nontriviality results for a number of specific extremal problems in Chapters 1 and 2. These results can be reduced to the proof of the fact that the set of data is dense in the corresponding function space in which the boundary value problem (0.2) has a solution.[1] Such a result is obtained in the following cases:

(a) $F(x, y)$ is the Cauchy problem for an elliptic equation and u is the Cauchy data;

(b) F is a mixed boundary value problem for the backward heat equation and u is the right-hand side of this equation.

Theorems in Chapter 3 about the approximate controllability of parabolic equations also belong to this group.

If equation (0.2) is nonlinear with respect to y, there are only several noncoordinated results in this direction. We present in details only one of them. Namely, we prove the density of the set of those right-hand sides in the corresponding space for which the mixed boundary value problem for the Navier–Stokes equations in a three-dimensional domain has a sufficiently smooth (consequently, unique) solution (cf. [**52, 53**]). Perhaps, this result is more interesting in the theory of unique global solvability of the Navier–Stokes equations than in the theory of extremal problems.

Chapters 4 and 5 are devoted to optimal control problems for incompressible viscous fluid described by the Navier–Stokes equations. In Chapter 4, we study an optimal control problem with distributed control, where the control is the right-hand side of the Navier–Stokes equations (i.e., the density of the external force). The physical problem is to accelerate the fluid to a prescribed velocity with the minimal work of external forces. We give necessary and sufficient conditions on the data of an extremal problem under which a solution of this problem exists. We also obtain necessary and sufficient conditions for a minimum in this problem which, together with the optimality system and the necessary second order condition for a minimum, contain some inequality that takes into account the global structure of the problem. Note that a large parameter M naturally arises in this problem. We show the uniqueness of a solution to the extremal problem for sufficiently large M. This result is the most essential one in Chapter 4. In author's opinion, the ideas of the proof of this result can be successfully used in the study of uniqueness of solutions in many important applications of the optimal control problems. We also obtain the asymptotics of solutions to the optimal control problem as $M \to \infty$.

In Chapter 5, we study the minimization problem for the work needed to overcome the drag exerted on the body moving in a fluid. The minimization is achieved by controlling the velocity of the fluid at the boundary of the body. More precisely, we study a two-dimensional analog of this problem. This problem is easily reduced to the study of an optimal control problem for a two-dimensional fluid flow surrounding a two-dimensional bounded domain with control at the boundary of this domain. Here, the main results are the solvability of the extremal problem and the deduction of the optimality system. To establish these results, it is necessary to develop the theory of nonhomogeneous boundary value problems for evolution Navier–Stokes equations. This theory is described in Chapter 5.

The remaining two chapters deal with applications of the theory of optimal control of distributed parameter systems. In Chapter 6, we consider the Cauchy

[1]This result immediately implies the validity of the nontriviality condition for the set U_∂ with a nonempty interior. Note that this result is also used in the verification of the nontriviality condition for some sets U_∂ with empty interior.

problem for a second order elliptic equation in a conditionally well-posed formulation. A feature of this formulation of the above ill-posed boundary value problem is that there is continuum of solutions and, instead of the uniqueness theorem, we have to prove that any two solutions are close (which is done in Chapter 6). Further, for the constructive solution of this problem, it is necessary to construct one of these solution, a quasisolution. A quasisolution is defined as a solution to a certain extremal problem. Using the theory of extremal problems developed in the previous chapters, we construct a quasisolution and investigate some properties of it.

In Chapter 7, we study the local exact controllability of the Navier–Stokes equations in a three-dimensional bounded domain Ω. We consider the boundary control, as well as the distributed control concentrated in a subdomain of Ω.

Let $\widehat{v}(t,x)$, $t \in (0,T)$, $x \in \Omega$, be a given solution to the Navier–Stokes equations (without control), and let $v_0(x)$ be the initial value that is sufficiently close to $\widehat{v}(0,\cdot)$ in the corresponding norm. The local exact controllability problem consists in choosing a control in a given class such that the solution $v(t,x)$ to the Navier–Stokes equations corresponding to this control and starting with the initial data v_0 coincides with $\widehat{v}(T,x)$ at $t = T$. The local exact controllability guarantees the possibility of stabilization of a nonstationary solution $\widehat{v}(x)$ by using a control of the given class.

The proof of the local exact controllability reduces to verification of the same property for the linearized Navier–Stokes equations. Hence we can reduce the case under consideration to the study of an ill-posed boundary value problem for these equations. To regularize this boundary value problem, we write an optimality system for a certain optimal control problem. The solvability of this optimality system can be established by using some Carleman estimates.

Thus, the theory of optimal control is used in Chapter 7 in the proof of properties of the local exact controllability.

Each chapter is supplied with short bibliographic comments. The literature devoted to optimal control problems is huge, and these comments do not pretend to be complete. We mainly indicate references that are close to the topics of the corresponding chapter.

A few words about our notation. Formulas, assertions, etc., are numbered separately in each chapter. For example, formula (5.2) is formula (2) in §5 of the current chapter. In references to formulas, assertions, etc. from other chapters, we add the number of the chapter in the notation. For example, formula (3.5.2) means formula (5.2) in Chapter 3.

Sometimes, we need to refer to a certain relation within a numbered formula. In this case, we use a subscript to indicate a particular relation in the formula. For example, if, say, formula (3.2) contains several relations and we want to refer to the third relation there, we write (3.2_3).

Acknowledgments. My interest in the subject of this book was initiated, in particular, by my work at the Chair of general control problems in Mechanics and Mathematics Department of Moscow State University. Discussions on the theory of optimal control with my colleagues and, first of all, with Professor V. M. Tikhomirov, were very useful. To all of them I express my deepest gratitude.

I am happy to thank Professor M. Gunzburger, the Head of Mathematical Department of Iowa State University, for the excellent opportunity to work on this book during my visits in 1997 and 1999.

I am deeply grateful to Tamara Rozhkovskaya, Editor-in-Chief of Scientific Books (Novosibirsk), for resolving numerous problems occurring during the preparation and publication of the manuscript. Without her support this book would have never been published.

I express warm thanks to my wife Irina Fursikova for constant support and patience.

The research presented in this book, was partially supported by the Russian Fund for Fundamental Research (grants Nos. 96-01-00947 and 96-15-96072).

Andrei Fursikov

June 1999, Moscow