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Metric Characterization of Random Variables and Random Processes

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Preface

A usual tool for solving various problems in probability theory and in the theory of stochastic processes defined on a probability space $(\Omega, \mathfrak{F}, \mathsf{P})$ is to introduce a space or a class of random variables characterized by the existence of certain moments or by the rate of growth of the moments, by the behavior of semi-invariants, by inequalities for the moment generating functions, or by the form or analytic properties of the characteristic functions. The following examples of spaces of random variables are well known: $L_p(\Omega)$ spaces, $p \geq 1$; Orlicz spaces; the class of Gaussian random variables and the class of sub-Gaussian random variables; the class of random variables satisfying the Cramér condition; the class of random variables satisfying the Bernstein condition; etc. The introduction of a particular class of random variables always has its own history and depends on many reasons, where the factor of importance, and sometimes of the utmost importance, has to do with the inequalities for "tails" of the distribution of a random variable belonging to this class or of a sum of random variables in this class.

In this context, it is appropriate to quote S. N. Bernstein [**Bernstein, 1964a**]:

> *... a proper application of the classical Chebyshev's argument enables one to obtain an inequality much more precise than the Chebyshev inequality, if one admits certain restrictions which are normally satisfied in practice.*

Inequalities of this type for different classes of random variables and their applications in the theory of large deviations, in particular, inequalities estimating probabilities of large deviations, were and still remain an object of intensive study (see, for example, [**Ibragimov and Linnik, 1965**], [**Petrov, 1972, 1987**], [**Saulis and Statulevičius, 1989**], [**Yurinsky, 1995**], and the bibliography therein).

It is remarkable that many traditionally considered classes of random variables are linear spaces, and even Banach spaces, and the characteristics of a random variable determining whether the variable belongs to a certain class are directly related to the norm in this space. Usually the smaller the space where a random variable lives, the better the order of decay of the estimate of "tails" of the distribution of this variable. This enables us to apply methods of functional analysis, in particular, the theory of K_σ-spaces and the theory of Orlicz spaces, to the study of behavior of a sum, or of the maximum, of a finite family of random variables belonging to the same class of random variables. This approach is especially successful for families of random variables whose "strong" norm is comparable to a "weaker" norm. A well-known example is that of Gaussian families of zero-mean random variables having the sub-Gaussian norm equal to the mean square norm.

The method of "imbedding" of a family of random variables into an appropriate Banach space, or into a metric space, of random variables is also widely used in the theory of stochastic processes. Suppose that T is a parameter space and $X = (X(t), t \in T)$ is a stochastic process such that for each $t \in T$ the random variable $X(t)$ belongs to a Banach space $B(\Omega)$ of random variables. The geometric structure of the set $(X(t), t \in T)$ considered as a subset of $B(\Omega)$, in particular, the entropy characteristics of this set, are closely related to the fact that almost all (with respect to the measure P) sample paths of the process X are bounded. If the parameter space T is supposed to be a topological space, for example, a metric space, then the same characteristics contain information on the continuity of almost all sample paths of the process X, on their moduli of continuity, etc. In this case it is usually assumed that either the process X is separable or a stochastically equivalent modification of X is considered. Observe that the entropy approach makes it more appropriate to consider entropy characteristics of the parameter set T with respect to the induced deviation $r_X(t,s) = \|X(t) - X(s)\|$, $t, s \in T$, where $\|\cdot\|$ is the norm or a similar functional on the space $B(\Omega)$, than to use entropy characteristics of the set $(X(t), t \in T)$ in the space $B(\Omega)$.

If estimates of the probability of large deviations can be obtained for random variables regarded as elements of the space $B(\Omega)$, then close estimates usually hold for the probability of large deviations of the supremum of the process X under the assumption that X is almost surely bounded.

The entropy approach to the study of properties of Gaussian processes was developed by R. M. Dudley, V. N. Sudakov, X. Fernique, and other mathematicians. Entropy methods are useful when applied not only to Gaussian processes but to different classes of processes often appearing in applications: sub-Gaussian processes, pre-Gaussian processes, Orlicz processes. Here it is important not simply to choose the smallest possible space $B(\Omega)$ into which the stochastic process is to be embedded but also to find an appropriate $B(\Omega)$-norm and estimates of this norm calculated for a particular value or for an increment of the process in terms of weaker norms or other characteristics of the process. The properties of general Orlicz processes were studied by G. Pisier, X. Fernique, C. Nanopoulos, P. Nobelis, M. Ledoux, M. Talagrand, and other authors.

The topic covered in this book is the study of metric and other close characteristics of different spaces and classes of random variables and the application of the entropy method to the investigation of properties of stochastic processes whose values, or whose increments, belong to given spaces. The spaces of random variables considered in this book contain both general K_σ-spaces or Orlicz spaces and their subspaces of special form. Traditional $L_2(\Omega)$-processes, Gaussian processes, and functional series with independent random terms do not play a particular role in the book and appear rather as examples illustrating the general theory. On the other hand, the following processes are considered in more detail: pre-Gaussian processes, shot noise processes representable as integrals over processes with independent increments, quadratically Gaussian processes and, in particular, correlogram-type estimates of the correlation function of a stationary Gaussian process, jointly strictly sub-Gaussian processes, etc.

Many works are devoted to probability metrics and their applications to the theory of summation of independent random variables, among which the book [**Zolotarev, 1986**] should be specially mentioned. The topics of some chapters of our book have natural intersections with the books [**Saulis and Statuliavičius, 1989**] and [**Ledoux and Talagrand, 1991**]. The contents and technical tools involved in this book are closely related to authors' research and their scientific interests. There are many challenging problems and results beyond the limits of the book. However, the authors hope that the problems and methods presented in this book will be interesting to the reader and will show the interrelation of probabilistic methods, analytic tools, and functional approach.

The book consists of eight chapters which can conditionally be divided into four parts. The first part contains the first two chapters and deals with classes of random variables and their metric characteristics. We want to note that metrics and norms considered here need not be norms and metrics in the commonly accepted sense.

Chapter 1 deals with the space of sub-Gaussian random variables, the space of pre-Gaussian random variables, and subclasses of these spaces containing strictly sub-Gaussian or strictly pre-Gaussian random variables, respectively. Different characteristics of these random variables are considered: sub-Gaussian standard, moment functionals, semi-invariant functionals, etc. Special attention is devoted to inequalities estimating "tails" of the distribution of a random variable, or of a sum of random variables, in the above-mentioned spaces.

In Chapter 2, general Banach spaces of random variables are studied, first of all K_σ-spaces and Orlicz spaces. Majorant characteristics of these spaces are introduced, enabling us to obtain inequalities for the distribution of the maximum of a finite number of random variables belonging to these spaces. Exponential Orlicz spaces and their subspaces are considered. The relation between these spaces and those considered in Chapter 1 is established. Inequalities for probabilities of large deviations are given.

The second part of the book includes Chapters 3 and 4 and deals with properties of stochastic processes "imbedded" into a space of random variables considered in the first part.

Chapter 3 is devoted entirely to the study of properties of stochastic processes whose values are random variables in a certain (general or more concrete) K_σ-space or in an Orlicz space. Entropy characteristics of the parameter set calculated with respect to metrics induced on this set are used to obtain estimates for the distribution of the supremum of the underlying process. Conditions providing that sample paths of a process are almost surely bounded or almost surely continuous are established. Moduli of continuity of sample paths of a process are considered. Examples are used to illustrate how to use general statements in order to obtain the Kolmogorov theorem, the Hunt theorem, the Dudley theorem, and a number of other well-known results.

Chapter 4 is devoted to pre-Gaussian processes. General results in Chapter 3 hold for these processes. However, the semi-invariant premetric induced by a pre-Gaussian process on the parameter set is a natural characteristic of this process. The properties of this premetric are very different from those of a metric. For example, this metric does not satisfy the triangle inequality. Still, one can define entropy characteristics of the parameter set with respect to this premetric, making it possible to obtain more subtle conditions for the continuity of sample paths of a pre-Gaussian process, conditions for the density of a family of pre-Gaussian processes in the space of continuous functions, etc.

The third part of the monograph contains Chapters 5, 6, and 7 and deals with applications of the general theory.

Chapter 5 is devoted to shot noise processes serving as a mathematical model of various physical phenomena. Conditions for the continuity of sample paths of these processes are established, and exponential estimates for the distribution of the supremum are obtained. Conditions for the asymptotic normality of a pre-Gaussian shot noise process in the space of continuous functions are also given. A similar range of problems is solved in Chapter 6 for correlogram-type estimates of the correlation function of a stationary Gaussian process.

Chapter 7 deals with jointly strictly sub-Gaussian stochastic processes. Lévy–Baxter type theorems are established, and the rate of convergence is obtained for a narrower class of processes, jointly pseudo-Gaussian processes. The boundary value problem for a homogeneous hyperbolic equation whose initial conditions are jointly strictly sub-Gaussian random functions is also considered in this chapter. The background for the Fourier method as applied to this problem is given.

The fourth part of the book contains Chapter 8, where the necessary auxiliary material is outlined. References to the literature used by the authors in writing this book are given in Comments and References.

The proofs of all basic statements are presented in full or at least sketched. A small number of statements which are less important for the basic text are given without proofs, but these proofs can be recovered by the interested reader. Different sections of this book were used as a basis for advanced courses in probability theory and theory of stochastic processes given by the authors in Kyïv Taras Shevchenko National University and National Technical University of Ukraine "Kyïv Polytechnic Institute".

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