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Hyperbolic Partial Differential Equations and Wave Phenomena

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Preface to the English Edition

This book is a translation into English of a book that I wrote in Japanese. This book is based on my lectures at various universities.

When I originally set pen to paper, I had only a Japanese audience in mind. So, I am deeply pleased and honored that by means of this English translation the material in this book will reach a wider audience.

I would like to express my appreciation to the American Mathematical Society for publishing this translation. My heartfelt thanks go to Bohdan I. Kurpita for his translation of my book from Japanese into English.

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Mitusuru Ikawa

Preface to the Japanese Edition

In this book we discuss initial boundary value problems for second order hyperbolic equations. The term hyperbolic equation refers to members of a specific class of partial differential equations. The most representative examples of this class are the partial differential equations that describe wave phenomena. In essence, the study of hyperbolic equations and the mathematical investigation of wave phenomena can be thought of as one and the same thing. For the purposes of this book, we will restrict our attention to linear equations. What this means for wave phenomena is that the oscillations cannot be particularly large.

Most likely, the first image the mind conjures up when one hears or reads the word "wave" is of water lapping back and forth. More precisely, the surface of the water undulates in a periodic manner with the effect being passed on to the surroundings. This type of phenomenon can also be seen when a string vibrates or a membrane oscillates.

Another form of "wave" behaviour that readily comes to mind is that of sound propagating through air or some other substance. Another example is an electromagnetic wave. A less sanguine example is that of an earthquake, which is a wave that radiates outwards from an epicenter and, on occasion, the subsequent oscillations have a devastating effect causing widespread destruction and bringing misery and death.

As the above examples illustrate, we encounter many wave phenomena on a daily basis, some wave phenomena can be readily seen and some not. At first sight, these wave phenomena seem disparate since they arise from different physical circumstances, however, if we write the mathematical formulae that govern these phenomena, then the phenomena all conform to partial differential equations of the same form; namely, *hyperbolic equations*.

The added benefit of expressing these types of phenomena mathematically is that investigating the partial differential equations leads to detailed insight into wave phenomena. In addition, the similarity among the partial differential equations for various wave phenomena suggests a deep commonality between the phenomena.

In the case of the equation for the propagation of sound, the unknown function is a scalar function. On the other hand, in Maxwell's equations, which govern the transmission of electromagnetic waves, the unknown function is vector valued. This implies that even though the hyperbolic equations are of the same type, the complexity of the associated phenomena will differ according to whether the unknown function is scalar valued or vector valued. Thus, by focusing our deliberations on the characteristics of each of the equations, we will unearth properties of both sound and electromagnetism.

Throughout this book, linear hyperbolic equations of second order with a scalar-valued unknown function will be central to our discussions. The reason is that this type of equation is common and fundamental for all of the equations that describe wave phenomena. With this in mind, our overarching aim is to consider particular linear hyperbolic equations of second order and elucidate the properties of the phenomena governed by this second order equation. In this way, we will discover properties that are common to various wave phenomena.

In summary, we will concern ourselves with a wave transmitted in a space with boundary, and, having been given the initial condition(s) and the state on the boundary, we will investigate how the solution develops with time. Towards this end, we first need to give a mathematical proof of the existence of a solution for the given problem. Then, our core problems are to clarify mathematically observations such as the direction of motion, the reflection of the wave at a boundary, the refraction of the wave at the surface of two substances that are in contact, etc. The clarification of these observations will be obtained by determining properties of the solution.

In closing, I wish to express my deep gratitude to the various members of the editorial board who encouraged me to write this book. In particular, I would like to extend my heartfelt appreciation to Professor Aomoto who read the original manuscript and offered many helpful suggestions.

Also, I wish to thank the editorial staff of the publisher, Iwanami. I have great admiration for all their efforts.

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Mitusuru Ikawa

Outline of the Theory and Objectives

As explained in the preface, the primary objective of this book is to understand wave phenomena in mathematical terms.

So immediately in Chapter 1, we develop the mathematical laws that govern the behaviour for several wave phenomena. To be more precise, the mathematical expressions that we derive in this first chapter consist of a relationship or relationships between the partial derivatives of the unknown function(s); such a relationship(s) is usually referred to as a *partial differential equation* (*or system of equations*). Among the phenomena to be considered in Chapter 1 are the vibration of a string, the oscillation of a spring and the propagation of sound.

As illustrative examples, we look at the vibration of a string and the oscillation of a spring. In the case of a string, if the string is disturbed from its state of rest, then the tension of the string causes a force to arise that pulls the disturbed part towards its rest position. In fact, this force gives a certain acceleration to the string, and the vibration is passed on to the surroundings. Thus a wave is formed.

While for the case of a spring, instead of tension we need to investigate the effect of elasticity. Briefly, what we mean is that when we apply a force to some part of the spring, we effect a contraction on this part of the spring, which, in turn, passes an oscillation to the surrounding parts.

Even the casual observer should be able to perceive a certain dichotomy between the nature of the vibration and the oscillation described above. In the case of the vibration of a string, each part of the string vibrates perpendicular to the direction in which the wave moves. In other words, each part of the string vibrates in a direction that is perpendicular to its position at rest, and with time the vibration is transmitted along the stretched string in the form of a wave. Hence, this kind of wave is known as a *transverse wave.*

In contrast, each part of a spring oscillates in the direction of the extended spring. So, this kind of wave is called a *longitudinal wave.*

From the discussion above, it is clear that the respective waves arise from quite different physical conditions, and the vibration and oscillation, themselves, are substantially different. However, as we will see in Chapter 1, if we derive the respective partial differential equations for these two phenomena, what we see is that the differential equations are exactly the same. This means that even though the vibration/oscillation are perceived differently, because these two waves have the same equation, a property for one of the waves also becomes a property of the other wave. For example, the speed of propagation to its surroundings is constant for each of them, reflection at its end point occurs for each of them, etc.

Continuing on from the introduction of the equation of the wave for the string and spring, we next turn our attention to the equation for the propagation of sound. Even if the dimension of the space is high, the astute reader will not be surprised to learn that the partial differential equation which will be found has the same form as that for a string or a spring. Therefore, an immediate consequence is that the propagation of sound must share the same properties, two of which were just mentioned, as the vibration of a string and the oscillation of a spring.

Having established the above, we will next write down, but not derive, *Maxwell's equations* that describe the propagation of electromagnetic waves and *elastic waves* that describe how a wave is transmitted through an elastic body, one example of such a wave comes from the observation of earthquakes. In both of these cases, the equations have unknown functions that are vector valued. Therefore, the oscillating phenomena that are governed by these equations include rather complicated terms. However, by careful consideration of these equations, it is possible to see that they consist of several equations each of which gives an expression as for the propagation of sound, and so without equivocation we can call them *wave equations.* In fact, if we restrict ourselves to relatively simple conditions, then we are able to see that each part of the unknown function becomes a solution of the wave equation. Conversely, by using the solution of the wave equation, we can construct solutions of the above equations. In the above sense, as stated in the Preface, the study of a single hyperbolic equation forms a basis for the investigation of a wave.

Also in Chapter 1, we define what we mean by a linear hyperbolic equation of second order. Then, in Chapter 2, we discuss the basic properties of the solution of this equation. As one of the more prominent properties of the solution, we will find that the speed of propagation of the solution is finite. For the wave equation we know that the solution propagates with a fixed speed, and thus we can conclude that a common property of hyperbolic equations is that the speed with which the oscillation/vibration, given at some part, propagates to its surroundings is finite.

Having completed the above, we will prove that the solution for the initial boundary value problem exists and is smooth.

In Chapter 3, we consider the asymptotic solution of the initial boundary value problem for hyperbolic equations. As stated above, in Chapter 2 we prove the existence of the solution. The proof depends on methods from functional analysis. However, when the state is fixed, these methods are not very suitable for studying the detailed behaviour of the solution. On the other hand, using the methods of asymptotic solutions, we can explicitly construct a good approximation to the solution, the existence of which has already been guaranteed. Thus, we can investigate in detail the way the solution propagates under some particular condition for one of the phenomena governed by the equations. In Chapter 3 we look at the propagation of a particularly important wave with a high frequency.

Light is a good example of a high frequency wave. If we take a minute to consider everyday occurrences, then what stands out is that light travels in a straight line and, also, when it hits a boundary it is reflected. Another light phenomenon that probably is familiar to us from experience is the refraction of light that occurs at the boundary of air and water. Properties such as those just mentioned for light (i.e., a slew of properties of waves that are commonly, if unconsciously, observed by us) in fact, can be explained mathematically by making use of the asymptotic solution.

By similar means, when the solution has a discontinuity, we can study how this discontinuity is transmitted.

In Chapter 4, with respect to the decay of the local energy, we will investigate the propagation of a wave outside a bounded obstacle. If there is no loss of energy due to, say, friction, then the total energy for the wave does not change with time. However, the positions that bear the energy, i.e., the positions of vibration/oscillation, will gradually recede into the distance with the passage of time.

Now, suppose that an observer is standing at some particular point. With the person stationary, let us assume that a vibration or oscillation arises somewhere and subsequently this leads to the transmission of a wave. Further, this wave passes by the observer and on hitting some solid object the wave is reflected repeatedly. After it has passed by the stationary observer a number of times, the effect, with time, will begin to diminish, and eventually in the environs of the observer the waves will die out. In fact, the consideration of such a phenomenon is the same as the investigation of the decay of local energy.

The reader should keep in mind that the problems that we consider in Chapters 3 and 4 relate only to very particular characteristics of waves. In contrast to a wave with a high frequency, a wave with an extremely low frequency, that is, one with a very high wavelength, has virtually no aftereffect even if it does hit some obstacle. Thus, what we can say is that the wave passes by without any reflection. In this book, we will not consider such a phenomenon. Also, we will not cover how the obstacle's shadow effects the wave. This problem is quite a difficult one. However, from the perspectives of both mathematics and physics, this problem is a very interesting one and worthy of future research.

If we try to encapsulate the formal aims of this book, then, first, it is to prove in the strict mathematical sense the existence of a solution, and, in addition, it is to describe the basis of the mathematical means that explain the most typical properties of a wave. To aid the reader's comprehension of our fulfillment of these aims, we will on several occasions repeat similar arguments.

Nomenclature used in this book

We denote the n-dimensional Euclidean space by $\mathbb{R}^n$ and a point in $\mathbb{R}^n$ by $x = (x_1, x_2, \ldots, x_n)$ with $x_j \in \mathbb{R}$ $(j = 1, 2, \ldots, n)$.

(1) Suppose D is an open set in $\mathbb{R}^n$. Then we will write the partial derivative of first order of a function f defined on D as

$$\frac{\partial f}{\partial x_j} \quad \text{or} \quad f_{x_j}.$$

For the exponent $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with $\alpha_j \in \{0, 1, 2, \ldots\}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$, the partial derivatives of higher order,

$$\frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}},$$

will be denoted by

$$\partial_x^\alpha f(x) \quad \text{or} \quad f^{(\alpha)}(x).$$

(2) We denote by $C(D)$ the space consisting of all the continuous functions that are defined on D. Then, for $m = 0, 1, 2, \ldots$ we define $C^m(D)$ by $C^0(D) = C(D)$ and otherwise by

$$C^m(D) = \{f \in C(D);\ \text{for all } |\alpha| \leqslant m \quad f^{(\alpha)} \in C(D)\}.$$

Further, we set $C^\infty(D) = \bigcap_{m=1}^{\infty} C^m(D)$.

(3) We set $\mathcal{B}(D) = \{f \in C(D); f \text{ is bounded in } D\}$; then for $m = 1, 2, \ldots$ we define

$$\mathcal{B}^m(D) = \{f \in C^m(D);\ \text{for all } |\alpha| \leqslant m \quad f^{(\alpha)} \in \mathcal{B}(D)\}.$$

In addition, we set $\mathcal{B}^\infty(D) = \bigcap_{m=1}^{\infty} \mathcal{B}^m(D)$.

(4) Let E be a linear space. Then $C^m(D; E)$ denotes all the functions defined on D that take their values in E and are functions with continuous partial derivatives of mth order with respect to the topology on E.

(5) We will denote Sobolev spaces by $H^m(D)$ $(m = 0, 1, 2, \ldots)$; a definition of these spaces is given in §2.2(a).