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Simplicial and Operad Methods in Algebraic Topology

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Preface

In recent years, in solving various problems of algebraic topology and, in particular, difficult problems of homology theory and homotopy theory, the usage of algebraic structures more complicated than just a topological monoid, an algebra, a coalgebra, etc. has been required more and more often.

Among the first examples of these structures were the Massey products introduced by W. S. Massey in 1954, which are partial, multivalued, and many-placed operations. The Massey products naturally arise on homology of differential algebras; they are used for computing differentials of various spectral sequences.

Another example of many-placed operations is provided by the A_∞-structure, which was introduced by J. D. Stasheff in 1963 with the goal of describing loop spaces. He proved that a connected CW-complex possesses an A_∞-structure if and only if it is homotopy equivalent to a loop space.

A general method for describing many-placed operations on topological spaces was proposed by J. P. May in 1972. He introduced the concept of an operad $\mathcal{E}$ as a family of topological spaces $\mathcal{E}(j)$, $j \geq 0$, whose points should be thought of as abstract j-ary operations. An action of an operad $\mathcal{E}$ on a topological space X is a family of operations $\mathcal{E}(j) \times X^{\times j} \to X$. By restricting these operations to points of the spaces $\mathcal{E}(j)$, we obtain many-placed operations $X^{\times j} \to X$ on X.

May showed that on any n-fold loop space there is an action of the little n-cubes operad E_n, and each arcwise connected E_n-space has the weak homotopy type of an n-fold loop space.

The concept of an operad turned out to be useful not only in describing structures on topological spaces, but also in investigating various algebraic structures. In 1981 the author generalized the notion of operad to the case of the category of chain complexes. It was shown that the singular chain complex $C_*(X)$ of a topological space X admits an action of the operad E_∞, and, in the case of a simply connected space, this action determines the weak homotopy type of this space. Thus many problems of homotopy theory can be reduced to the study of the E_∞-structure on the chain complex.

In this book we present a detailed study of the concept of an operad in the categories of topological spaces and of chain complexes. We introduce the notions of an algebra and a coalgebra over an operad and investigate their properties. We elucidate the algebraic structure of the singular chain complex of a topological space and show how the problem of homotopy classification of topological spaces can be solved using this structure. For algebras and coalgebras over operads, we define standard constructions, in particular, the bar and cobar constructions. We apply operad methods to computing the homology of iterated loop spaces, investigating the algebraic structure of generalized cohomology theories, describing cohomology of groups and algebras, computing differentials in the Adams spectral sequence for the homotopy groups of the spheres, and some other problems.

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