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Geometry of Characteristic Classes

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Preface

The purpose of the present volume is to give expositions on the basics of some new theories concerning geometry of characteristic classes which have appeared since the end of the 1960's.

Among characteristic classes, there are Stiefel-Whitney classes, Euler classes or Pontrjagin classes and Chern classes as typical examples. These classes were introduced during the 1930's and 1940's, and since then they have played fundamental roles in the classification as well as analysis of the structure of manifolds.

On the other hand, from the end of the 1960's onward, there arose a few theories which treat finer structures of manifolds than before. These include Gel'fand-Fuks theory, Chern-Simons theory and also the theory of characteristic classes of flat bundles, which are closely related to each other. These new characteristic classes are defined when the above mentioned classical characteristic classes vanish (partially) so that sometimes they are called the secondary classes. Among other things, the characteristic classes of flat bundles have been shown to have intimate relations with not only geometry but also algebraic geometry and number theory. It is plausible that they will play more and more crucial roles in future mathematics.

The theory of characteristic classes of fiber bundles, whose structure groups are infinite dimensional groups such as the diffeomorphism groups of manifolds, remain largely unknown. However, in the case of diffeomorphism groups of surfaces, quite detailed studies have been made. This is the theory of characteristic classes of surface bundles which began in the 1980's.

As will be mentioned in "Directions and Problems for Future Research" at the end of this book, studies of the above new theories are continuing from various points of view. Although the description of this book is limited to the foundational part, the author would be happy if readers are interested in these theories.

The period when the subjects of this book were developed overlaps the thirty years since I began mathematical research in graduate school. I would like to express my gratitude to my teachers, superiors and colleagues for their support. Also I would like to express my thanks to the staff of Iwanami Shoten.

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Summary and Purpose of This Volume

As was already mentioned in the preface, we have Stiefel-Whitney classes, Pontrjagin classes and Chern classes as typical examples of characteristic classes. They were introduced one after another during the ten years after the middle of the 1930's, and foundations of the theory of characteristic classes of principal bundles, whose structure groups are finite dimensional Lie groups, were already established in the early 1950's.

There are two main methods in the theory of characteristic classes. One is topological and the other is differential geometrical. The former method appeared first. In fact, all the characteristic classes mentioned above were introduced first as the primary obstructions to the existence of sections of certain fiber bundles which are associated to various principal bundles. Next there arose a method to compute the cohomology groups of the classifying spaces, which are described in terms of Grassmannian manifolds, by means of explicit cell decompositions. On the other hand, the latter differential geometrical method describes the way principal bundles are curved by differential forms using the concepts of the connection and the curvature. This is called the Chern-Weil theory. Roughly speaking, the theory of secondary characteristic classes is a refinement of the latter method.

We summarize the contents of this book briefly. In Chapter 1, we give a short exposition of the de Rham homotopy theory. This theory was begun by Sullivan in the 1970's, and it enhances extensively the theorem of de Rham, which describes the cohomology of manifolds in terms of differential forms. Roughly speaking, it is meant to grasp information about homotopy types of spaces by differential forms rather than merely the cohomology groups. This theory is not directly related to characteristic classes. However, it is one of the basic tools for the study of manifolds, and it seems that it will develop further in the future. It is only in §3.5 where we shall use the ingredients of this chapter. Nevertheless, the general idea behind this theory has meaning throughout all the other chapters.

In Chapter 2, we consider characteristic classes of flat bundles. A flat bundle is a principal bundle equipped with a connection whose curvature vanishes identically. Hence, by the Chern-Weil theory, all the characteristic classes (with real coefficients) vanish. The structure of such a bundle can be described completely by a homomorphism from the fundamental group of the base space to the structure group which is called the holonomy group, or according to the context, the monodromy group. However, in general, it is very difficult to study this homomorphism directly. There arises then an idea to describe how the bundle is twisted in terms of cohomology classes. Since the curvature is identically zero, we can construct certain cohomology classes by using the connection form. More precisely, certain characteristic classes, called characteristic classes of flat bundles, can be defined based on cohomology theory of Lie algebras. We also give a quick introduction to the Gel'fand-Fuks cohomology theory, which can serve as characteristic classes of flat bundles whose structure groups are infinite dimensional.

In Chapter 3, we treat the theory of characteristic classes of foliations. A foliation is a certain striped pattern on a manifold, and its characteristic classes are certain cohomology classes of the manifold which describe global geometrical properties of the pattern. This theory is closely related to those of Cheeger-Chern-Simons and Gel'fand-Fuks, and it was established within a rather short period in the first half of the 1970's. It is a typical example of secondary characteristic classes. One characteristic of these classes is the following fact. Namely, in contrast with the classical characteristic classes which are cohomology classes with finite or integral coefficients, characteristic classes of foliations become cohomology classes with essentially real coefficients. This fact appears directly in a celebrated theorem of Thurston which showed that a characteristic class of foliations, called the Godbillon-Vey class, can vary continuously. In §3.5 we shall formulate an open problem which is derived from the above fact.

The subject of Chapter 4 is the theory of characteristic classes of surface bundles. The theory of characteristic classes of fiber bundles, whose fiber is a general manifold in a general dimension, is far from being understood because of difficulties in treating diffeomorphism groups which are infinite dimensional. However, in the 2-dimensional case, namely in the cases of surfaces, an exceptional phenomenon occurs which is caused by special properties of geometries on surfaces. More precisely, it turns out that surface bundles, which are objects of topology, have an intimate relationship with the moduli space of compact Riemann surfaces or the Teichmüller space, which belong primarily to algebraic geometry or complex analysis, so that we can develop a far richer theory than other dimensions. Here we shall give an introductory exposition of this theory.