Foreword

This book reproduces, without modifications, the "Doctor of Sciences" thesis of the remarkable mathematician Sergei Vassilievich Kerov (June 12, 1946–July 28, 2000). The thesis was written in 1992–1993 and defended in 1994. It is devoted mostly to analytical aspects of the asymptotic representation theory of symmetric groups and related problems of analysis.

I have to say several words about these topics, which appeared about 30 years ago. The theory itself and the name "Asymptotic representation theory" were suggested by me in the late sixties; this name refers to the whole complex of problems that lie at the interface of functional analysis, algebra, combinatorics, and probability theory, and concern the study of the behaviour of classical groups and their representations as the rank (degree) of the group tends to infinity.

It is natural here to single out two classes of problems: proper asymptotic problems and problems that concern the limiting object, i.e., an infinite or infinite-dimensional group which is an infinite analogue of a classical group. The principal example, which serves as a model for more difficult cases, is of course the asymptotic theory of symmetric groups and their representations. It was with this example that our activity started, just as in the 19th century the theory of finite groups and their representations had started with the symmetric groups.

An early example of an asymptotic problem of the first kind was considered in the early seventies in my papers with A. Shmidt [**20, 21**][1] on the limiting joint distribution of the normalized cycle lengths of uniformly distributed random permutations. Combinatorics is here closely intertwined with probability theory. Later, many papers were devoted to this topic, which has numerous applications (in number theory, coagulation schemes, populational genetics, etc.). This topic is considered partially in Chapter 4 of Kerov's thesis; it is also related to his later papers on the Poisson–Dirichlet measures.

I formulated the dual asymptotic problem for symmetric groups in the early seventies, and it was solved in joint papers with Kerov [**12, 18**]; this was the problem about the limiting shape of Young diagrams with the Plancherel statistics, or the typical representation of the symmetric group of high degree. From the technical point of view, it belongs to the class of "limit shape problems", which has recently become very popular. Namely, the problem asks what is the asymptotic behaviour of a configuration (for example, a Young diagram) that grows randomly but according to certain rules. This topic is developed in Chapter 3 of the thesis, where, in particular, the central limit theorem for the Plancherel measure is proved, and numerous relations to other problems of analysis are considered. For further development see, e.g., [**A.28, A.27, A.36, A.41, A.45, A.7, A.8, A.25, A.26**].

[1] A reference such as [**A.n**] refers to the additional reference list given at the end of this volume. A reference without A refers to the reference list in Kerov's thesis.

A survey of a more general class of limiting shape problems can be found in [**A.52**] and in the papers by A. Okounkov on Schur measures, e.g., [**A.38**].

It was clear from the very beginning that the problem concerning the asymptotic behaviour of representations and characters is of fundamental importance. However, only much later, in the nineties, did it become clear that this problem has many relations to other topics: spectra of random matrices (see, e.g., [**A.50, A.3, A.4, A.29, A.30, A.15**]), free probability theory (see, e.g., [**A.5–A.10, A.48, A.49, A.14**]), integrable problems and algebraic geometry (see, e.g., [**A.2, A.38, A.40, A.42, A.43**]), the theory of z-measures (see, e.g., [**A.13, A.39**]). This list is far from complete. Perhaps, one relation is worth special mention. I mean the solution of the so-called Ulam's problem on the longest increasing subsequence in a sample of n independent random variables uniformly distributed on an interval (or the longest increasing subsequence in a random permutation). This problem has a long history; an important role here is played by the RSK algorithm which interprets the length of this subsequence as the length of the first row of a random Young diagram. The problem was solved completely in our paper [**12**], where we proved the old conjecture that the answer is $2\sqrt{n}$. Note that the abovementioned theorem on the limit shape of a Young diagram, which was obtained in our work and, independently, by Logan and Shepp [**138**], and which gives complete information on the entire monotone structure of a random sample, is, however, not sufficient to find the asymptotics of the length of the first row, since this theorem gives only a lower bound on this length. So it required an important, significant technical observation related to Young tableaux, which was made by Kerov and completed the solution of Ulam's problem. Our proof of the general limiting shape theorem was based on a continuous analogue of the hook-length formula, estimations of probabilities, and solution of an integral equation; it was different from the proof in [**138**] and allowed us also to obtain estimates for the asymptotics of the maximum dimension of representations of symmetric groups. For an elementary proof of Ulam's problem, see, e.g., [**A.1**].

Problems of the second kind concern the study of the infinite analogue of symmetric groups, the infinite symmetric group of finite permutations. In this class of problems, there is a distinguished problem concerning the description of the characters of this group, which was solved by Thoma [**161**] by analytical techniques and which was considered from the new point of view in our paper [**13**], and solved by means of a very general ergodic method suggested in [**10**] (see the first chapter of the thesis). It can be naturally reformulated as a problem on the boundary of the Young graph.

In fact, this is a new class of problems which can be stated in purely analytical terms — as the description of harmonic functions of certain operators; in probabilistic terms — as the description of the Poisson or Martin boundary of random walks; in algebraic terms — as the computation of traces of AF-algebras; and, finally, in combinatorial terms — as the statistics of the number of paths and central measures in graded graphs. Although the ergodic method provides a general approach, analytical difficulties are different in each specific case, and they are far from easy to overcome. Our work on the computation of the characters of the infinite symmetric group by the ergodic method [**13**] was continued in papers by Okounkov [**54**], Olshanski and Borodin [**A.12**], and others, where several new proofs of Thoma's theorem were suggested and a number of other graphs were considered. The leading role here belongs to Kerov; he considered different branching parameters (see Chapter 1). The results in this field that were known by 1993 are collected in Chapters 1 and 2. Later this problem was intensely studied by Kerov for other graphs and measures on them (Chapter 2). For further results about central measures and boundaries of different graphs, see, e.g., [**A.24, A.21, A.22, A.23, A.19, A.20**]). In subsequent work, the ergodic method was repeatedly used for the description of invariant measures, characters, etc., see, e.g., [**A.46, A.44**].

A very important general conjecture by Kerov (see [**119**]) on the boundary of graphs related to the Hall–Macdonald polynomials is still open. In a subsequent paper, written by Kerov together with Olshanski and Okounkov [**A.34**], the solution is found for a special case of Kerov's conjecture, namely, for the so-called Jack graph, a version of the Young graph with different transition probabilities.

I think that the most interesting and original part of Kerov's thesis is the study of continuous Young diagrams. Already the first paper [**12**] on the limit shape of Young diagrams with the Plancherel distribution contained the transition from ordinary diagrams to continuous ones, and a continuous analogue of the hook-length formula. Kerov showed (see Chapter 4), first, that the limit shape of random Young diagrams with the Plancherel distribution appears naturally in a seemingly quite unrelated problem concerning separation of roots of orthogonal polynomials; and, which is even more important, that there exists a one-to-one correspondence between continuous diagrams and probability distributions that extends the correspondence between ordinary Young diagrams and their Plancherel transition probabilities; this correspondence is a nonlinear transformation of measures, which he rightly called the Markov–Krein transform. Thus Kerov linked the classical moment problem to the combinatorics of continuous Young diagrams. Ordinary diagrams correspond in this picture to discrete interlacing measures, and their Markov–Krein transform corresponds to the partial fraction expansion. Not less impressive is the relation, discovered by Kerov, of the Plancherel dynamics of continuous Young diagrams to a special solution of the Burgers equation: it turns out that the same limit shape of Young diagrams is a fixed point for this equation, and it attracts asymptotically the solutions of a certain class. And the most classical-looking result is the theorem that says that the interlacing roots of two adjacent orthogonal polynomials generate a Young diagram, and this diagram converges, as the number of the orthogonal polynomials goes to infinity, to the same limit shape. Kerov also found close relations of this topic to Voiculescu's free probability theory (the role of the Gaussian law in classical probability theory is played here by the semicircle law which is related to the same limiting curve) and to combinatorics (the hook walk and the interval shrinkage process). For further results concerning the combinatorial aspects, see, e.g., [**A.47, A.17, A.18**]. The results of Kerov on continuous diagrams and the Markov–Krein transform are set forth in more detail in his subsequent paper [**A.33**]. For further generalizations of the Markov–Krein transform, see [**A.35, A.51**].

During the intervening years the results of the thesis did not become outdated; they are as interesting to read now as they were at that time. Of course, the above-mentioned progress in the study of more subtle asymptotics and relations to random matrices and free probability could not be mentioned in the thesis; however, Kerov took an active part in this new research (see the list of his publications, at the end of this Foreword).

The thesis is written very clearly and can be regarded as a handbook for mathematicians who want to work in asymptotic representation theory and obtain information on many results of the first period of its development. The introduction, which occupies almost a quarter of the whole thesis, contains all definitions and main results, so that a reader who is interested only in statements will need to read only this extensive introduction.

The thesis includes only a part of the work done by Kerov up to 1994; the complete bibliography is given in this volume. I have asked G. Olshanski, A. Gnedin, and N. Tsilevich to prepare an additional list of references to the papers of subsequent years related to the topics touched upon in the thesis. Olshanski also prepared bibliographic comments, which are given at the end of the book. These comments cover only some of the topics that were studied in Kerov's thesis and developed later. The current state of this broad and rapidly developing area, asymptotic representation theory and its applications, demonstrates a very diverse and vivid picture.

Without any doubt, Kerov's work will be avidly studied by future generations of mathematicians.

As an addendum to this Foreword we reproduce an abridged version of the obituary which I wrote for the special volume "Representation theory, dynamical systems, combinatorial and algorithmic methods. VI", Zapiski Nauchn. Semin. POMI, v. 283, 2001, which was dedicated to the memory of S. Kerov; this obituary is translated by N. B. Lebedinskaya.

* * *

S. V. Kerov was a profound and original mathematician. He was gathering force not very rapidly, steadily advancing to more and more difficult problems and to the understanding of various mathematical connections. The list of his articles is not very large, but there are a number of serious papers, which will be studied and continued. His absolutely unexpected and sudden death did not allow him to complete several articles that were almost finished.

I was his scientific adviser when he wrote his graduation thesis and when he worked at his Candidate's (Ph.D.) thesis.* For many years, he was my colleague and coauthor; together we elaborated ideas that I had worked on earlier, as well as those that occurred to both of us later. At a later time, he became a prominent expert on combinatorics, the theory of symmetric functions, and many other fields. He carefully examined the literature, and I often learned new information from him. His own ideas and papers, especially in the last ten years, showed his deep insight into combinatorics, analysis, and probability.

Here I will try to outline his scientific path — I was a witness of much of it. In particular, I discuss in more detail our joint work, which began in the mid-seventies, continued with interruptions until the mid-eighties, and then recommenced in the late nineties.

From his third year at Leningrad University, Sergei participated in the seminar on dynamical systems, representations, and algebras that I have headed since 1967. His M. Sci. thesis was devoted to flows on nilpotent and solvable manifolds.

**Translation Editor's Note.* The Russian Candidate's thesis is equivalent to the Western Ph.D. thesis. The Russian Doctoral thesis, which this book is, is much more advanced, roughly equivalent to "first published book".

Sergei was interested in different areas of mathematics, and from the very beginning he was very earnest and thorough in his work and study. There were many talented students (Ya. Eliashberg, Yu. Matiyasevich, and others) of the same year as Sergei, but even at that time he had a high reputation, which was based on the impression of his unceasing internal activity. His modest and dignified manner was very attractive. In 1969, he became a post-graduate student at the Department of Analysis, and I was his scientific adviser. In 1969–70, I gave a course of lectures on C^*-algebras and related topics. It was a new topic to me, and I wanted to apply these techniques to dynamical systems and representation theory. Since representation theory had gradually become the main topic, the seminar was guided in major part by problems in this theory. Sergei studied the theory of C^*-algebras and the classical theory of representations of symmetric and finite groups. As a subject for his Candidate's thesis, I suggested the duality theory of $*$-algebras (the theory of "positive" duality). I put forward this idea generalizing the theory of Hopf algebras in 1971 and published a brief note on the geometry of states (the theory of packets) in 1972. The basic definition implied that for algebras in linear duality, multiplication in each of them (or multiplication and comultiplication) is an operation preserving positivity (but not multiplicativity as in Hopf algebras). The main problems were formulated, and Sergei tackled them with enthusiasm. In his Candidate's thesis, he carefully examined the finite-dimensional version, including the Plancherel duality and induction; he also studied the nontrivial commutative version (see [**A.31, A.32**]). This theory is not yet entirely known, but Kerov's two papers on this subject are often cited. Only recently, in papers with I. Ponomarenko and S. Evdokimov, we proved that the finite-dimensional algebras in Plancherel duality are just the so-called C-algebras in algebraic combinatorics. I have no doubt that these ideas will be used in the future. There are also many relations with hypergroups, the theory of generalized shifts, quantum groups, many-valued groups, and so on. Sergei studied B. M. Levitan's book thoroughly, and we discussed further applications of the theory of positive duality to differential equations.

By 1975, when his Ph.D. thesis was completed, I had drawn him to a new topic, asymptotic representation theory. We started by studying Thoma's work [**161**] on the characters of the infinite symmetric group (I learned about this paper from I. M. Gelfand and I. Segal) and with the problem concerning the asymptotics of the Plancherel measure on Young diagrams, which seemed to me to be of key importance. An important role in investigating these problems was played by our experience in ergodic theory and dynamical systems (ergodic method, invariant central measures, etc.). We worked a lot with AF-algebras and the K-theory of these algebras, and rediscovered some known facts. I think that our most important result in this field is the computation of the K-functor of the group algebra of the symmetric group [**128**].

Sergei's main research interest remained, however, related to questions concerning Young diagrams, symmetric groups, and related problems of analysis and combinatorics. These interests are well reflected in the present book.

Our plans included the study of asymptotic problems concerning representations of matrix groups over finite fields: this topic was outlined in the eighties, and we discussed it with A. Zelevinsky, who worked at that time on the application of the method of Hopf algebras to a somewhat different problem about the structure of the set of representations of all finite groups $GL(n, F_q)$. However, it was not until 1996 that we started a serious study of the characters of the groups $GL(\infty, F_q)$ and a new group GLB, [**A.54**] (see also [**A.53**]). I hope that this subject will be of top priority later on.

Starting in the mid-eighties, Sergei extended the range of his interests, writing a number of papers related to mathematical physics (see, e.g., [**41**]). After Jones' works on new knot invariants and Connes' report on them at the Bourbaki seminar, we showed that these invariants can be obtained by specialization of the traces of the infinite Hecke algebra, or the characters of the infinite symmetric group [**19**].

Even a detailed survey of Sergei's papers cannot cover the whole spectrum of his interests and the topics he thought about. A great deal remained incomplete, some of it only in drafts. Here I should mention Sergei's exceptional accuracy, his systematic character. More than thirty thick copybooks filled with detailed writings, complete with dates and titles, were left after his untimely death. Of course, something from these records had been published, but much remained in sketches. Sergei always took an active interest in the studies of other mathematicians in order to keep abreast of new developments. On the other hand, the ideas proposed by Sergei in discussions with young people often stimulated them in their work.

His pedagogical activity at the Department of Mathematics and Mechanics began too late (when he graduated from the university, he was not taken on the staff of the Department of Mathematics and Mechanics despite my strong recommendations). He first worked at the Computer Center, and later at some institutes that did not train professional mathematicians. Only in 1994 did he start to deliver lectures at the Department of Mathematics and Mechanics; for this reason, he had no time to bring up his own students, although his lectures on orthogonal polynomials, which were quite extensive and rich in content, were very popular. Nevertheless, he had young coauthors and followers in our country and abroad (see the list of his papers). His foreign scientific trips also began late, in 1991, but they were very successful and fruitful. By the early eighties, specialists were aware of our joint and of his own papers, and a number of very valuable invitations were received: Montreal, Ottawa, Harvard, Iowa, Rutgers, Paris, Kyoto, and so on.

In 2000–2001, special meetings dedicated to his memory were held at several conferences, in Berkeley, Cambridge (the Isaac Newton Institute), and St. Petersburg (POMI, and the Euler Mathematical Institute). High esteem and admiration shown by his colleagues, students, and friends were a natural tribute to this remarkable mathematician, a worthy and modest man. I find it hard to believe that Sergei is absent from our institute, our seminar, and conferences. His friends and colleagues, and the young people who knew him, all remember him with gratitude.

A. Vershik