Preface to the English Edition

This is the English translation of a book published in Russia in 2000. The book is a realization of our old plan to write a small book explaining the main ideas of integral geometry in the context of several simple examples, and it follows our point of view that even now certain specific examples play a bigger role in integral geometry than general results.

For these simple examples we selected the classical Radon transform, its generalization suggested by F. John, hyperbolic versions of the Radon transform, and the horospherical transform for the group $SL(2, \mathbb{C})$. In discussing the Radon transform, which, of course, is treated in other books, we emphasize several circumstances, which are usually not considered. One example is the projective invariance of the Radon transform. This allowed us to regard the affine Radon transform, the Minkowski–Funk transform, and the geodesic hyperbolic Radon transform as different realizations of the projective Radon transform. We also considered it important to illustrate, by simple examples, the central role played in integral geometry by the operator κ, which is responsible for the universality of explicit inversion formulas.

The English edition of the book contains some modifications and corrections. In particular, we added Chapter 5 devoted to integral geometry on quadrics, or, in other words, to a conformally invariant version of the Radon transform. This approach allows us to combine the previously mentioned version of the Radon transform with the hyperbolic horospherical transform.

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Preface

Integral geometry studies mainly integral transforms assigning to a function on a manifold X the integrals of this function over submanifolds that form a family M. It is assumed that the family M itself is endowed with the structure of a manifold. This establishes a correspondence between functions on the manifold X and functions on some manifold M of submanifolds of X. For instance, to functions on Euclidean space E^n one can assign the integrals of these functions over all possible lines; this rule defines an integral transform sending functions on E^n to functions on the manifold of lines.

Along with the integration of functions on X over submanifolds, integral geometry considers similar integral transforms of other analytic objects on X (densities, differential forms, sections of bundles, etc.). The main problems are in the description of the images and kernels of these transforms and in the construction of explicit inversion formulas recovering the original objects from their images. The first book devoted to this area of mathematics was the monograph by I. M. Gelfand, M. I. Graev, and N. Ya. Vilenkin [**8**].

Integral geometry interacts with the classical direction in geometry originating from Plücker, Klein, and Lie; the cornerstone of this direction is dualities between pairs of manifolds such that points of one manifold are realized as submanifolds of the other. Manifolds whose points have such geometrical nature, carry specific structures used in integral geometry. Transforms in integral geometry are studied in the language of these geometric structures. The typical examples of such manifolds are the four-dimensional variety of all lines in the three-dimensional projective space (the Plücker–Klein quadric) and, more generally, the Grassmann manifold of k-dimensional planes in the n-dimensional projective space, the Lie manifold of spheres, and so on. All these manifolds play a significant role in integral geometry.

The best known example of an integral geometry transform is the Radon transform which appeared in 1917 in Radon's paper [**31**] staying aloof from his other mathematical heritage. Radon considered the operator of integration over hyperplanes in Euclidean space and gave the exposition a surprisingly perfect form, which, uncommon at that time, combined analytic and geometric considerations and anticipated possible analogs of this transform in other homogeneous spaces. The attention was focused on the inversion formula reconstructing a function from its integrals over the hyperplanes. It is remarkable that the resulting formula is absolutely explicit and has principally different form depending on whether the dimension of the space is even or odd. In the odd-dimensional case the formula is local; namely, some differential operator is averaged along a family of parallel hyperplanes (and to reconstruct the function at a point, it suffices to know the integrals of this function only along the hyperplanes close to this point). In the even-dimensional case the inversion formula is nonlocal; namely, some integral operator is averaged along a family of parallel hyperplanes (in contrast to the differential operator arising in the odd-dimensional case), and we need integrals over distant planes. This corresponds to the fact that the Huygens principle holds for wave propagation in odd-dimensional spaces and fails in even-dimensional ones.

Already before Radon, Minkowski and Funk considered an analog of this transform for the spheres in which an even function on the sphere is reconstructed from the integrals of this function over the great circles. This reconstruction can be carried out by using spherical polynomials. Radon knew about these results, but he seemingly did not know that these transforms are projectively equivalent (see Chapter 1).

In 1938, F. John [**28**] considered a natural generalization of the Radon transform in which a function on three-dimensional space is integrated over all possible lines rather than over planes. A new exceptionally important feature of this construction is that the family of lines depends on four parameters. Thus, the John transform acts from functions of three variables to functions of four variables, and hence it is natural to expect that functions in the image satisfy an additional condition. The main observation of John was that these functions satisfy the ultrahyperbolic differential equation which completely describes the image. This observation, which was of great importance for the future of integral geometry, not only put overdetermined problems into circulation but also determined the relations between integral geometry and differential equations.

Starting from the 1940s, one of the central problems in mathematics was to develop an analog of the Fourier integral for noncommutative Lie groups. Among the first groups under investigation (mainly because of physical applications) was the Lorentz group $SL(2, \mathbb{C})$, i.e., the group of second-order unimodular complex matrices. For this group, I. M. Gelfand and M. A. Naimark succeeded in constructing a theory in which the role of exponential functions was played by irreducible infinite-dimensional unitary representations of the Lorentz group, and, at the same time, the main features of the classical Fourier integral were preserved. Obtaining analogs of the inversion formula and the Plancherel formula for the Fourier integral was the culmination of this theory.

Later on it became clear that the main point in the proof of these formulas was in the reconstruction of a function in $\mathbb{C}^3$ from the integrals of this function over the lines intersecting a hyperbola. Thus, a complex version of the John transform was considered; however, to recover the function, the integrals over the lines of the three-parameter family of lines intersecting the hyperbola were used rather than the integrals over all complex lines (forming a four-parameter family). This setting of the problem is natural because the reconstructed function depends on three variables. The analog of the Fourier transform on the Lorentz group is related to the above transform of integral geometry by an ordinary one-dimensional Fourier transform.

This observation opened new possibilities for integral geometry. First, the following problem arose: To what extent the phenomenon noticed in the case of Lorentz group is general? It turned out that, for a broad class of homogeneous spaces including complex semisimple Lie groups and Riemannian symmetric spaces, there is a transform of integral geometry, namely, the horospherical transform, which is related to an analog of the Fourier transform for these homogeneous spaces by means of the ordinary Fourier transform. This connection is similar to the relation between the Radon transform and the classical n-dimensional Fourier transform carried out by the one-dimensional Fourier transform. On the other hand, the context of integral geometry is much broader than that of representation theory. For instance, instead of the family of lines in $\mathbb{C}^3$ intersecting a hyperbola one may consider the family of lines intersecting any other algebraic curve and obtain a similar explicit inversion formula, although here there is no relation to groups or homogeneous spaces. This suggests the idea that perhaps the natural setting for multidimensional harmonic analysis contains not only groups or homogeneous spaces but also some more general geometric structures, in particular, "good" manifolds of submanifolds for which there are explicit inversion formulas.

It then seems natural to develop methods of geometric analysis meant for integral geometry and enabling one, in particular, to construct harmonic analysis on a broader class of homogeneous spaces including symmetric spaces. This became the mainstream for the further development of integral geometry. Much has already been done in this direction, but many things remain unclear. There is a rather complete picture for the problem of "good" (admissible) families of complex curves for which there is an explicit local inversion formula (see [**4, 1, 22**]). However, the results on families of complex submanifolds of dimension greater than one are far less complete (although the known results are sufficient to obtain the Plancherel formula for complex semisimple Lie groups by using the technique of integral geometry; see [**14**]). There are many problems that remain unsolved in the real case; this concerns especially the construction of nonlocal inversion formulas and the clarification of relation between the discrete series representations and integral geometry. Some results in this direction were obtained recently [**17**].

On the other hand, interesting and deep connections of integral geometry with multidimensional complex analysis, symplectic geometry, and nonlinear differential equations explicitly integrable by methods of the inverse problem were discovered; see [**19, 16, 12, 13**]. We also note that there are applied aspects of integral geometry, see [**2, 18, 30**]. The inversion of the Radon transform is the background of computer tomography. There are other physical problems (in astrophysics, geophysics, electron microscopy, etc.) in which the data can be interpreted as the Radon transform.

The aim of this book is quite modest. We intend to show some ideas and constructions of integral geometry by the example of the most elementary problems. We hope that the main part of the material is accessible to university students, who can see by these examples how the interaction of elementary analysis and geometry leads to beautiful and important results. The five chapters of this book are devoted to five particular transforms. We begin with the Radon transform (Chapter 1). An important point here (which is often neglected) is that the Radon transform is of projective nature, and in the projective version, it includes not only the affine Radon transform but also the Minkowski–Funk transform on the sphere. In Chapter 2 we study the John transform (the X-ray transform). It has already been noted that the inversion problem for this transform is overdetermined, and therefore it is of importance to understand how to pose this problem and describe a class of inversion formulas for appropriate three-parametric families of lines. This problem is solved here in the complex case. In Chapter 3 we present integral geometry on two-dimensional and three-dimensional hyperbolic spaces. The nonzero curvature of these spaces leads to a greater diversity of problems. In particular, there are two versions of the Radon transform, namely, geodesic and horospherical. However, the inversion formulas are similar to those in the Euclidean case, and their proof is only slightly more complicated. In general, it is typical for integral geometry that explicit inversion formulas have some standard structure and only slightly change if the geometry becomes more complicated (contrary to the formulas of group representation theory). This phenomenon has grave reasons, which have mainly been understood, but their discussion is beyond the framework of our book. We also consider the analog of the Fourier transform in the hyperbolic space; the results for this transform are obtained by using the horospherical transform. In Chapter 4 we develop integral geometry on the group $SL(2,\mathbb{C})$; as we have already mentioned, this example stimulated the modern development of integral geometry. Finally, in Chapter 5 we present a conformally invariant version of the Radon transform (the projective Radon transform). We give three variants of this transform, which are convenient in different applications; they are associated with the family of hyperplane sections of the sphere in $\mathbb{R}^{n+1}$, the family of hyperplane sections of the n-dimensional hyperboloid of two sheets, and the family of all spheres in $\mathbb{R}^n$, respectively. As in Chapter 2, we have an overdetermined problem of integral geometry and we give a universal inversion formula (the operator κ). Among interesting consequences is the conformal equivalence of the Radon transform and the horospherical transform in the hyperbolic space.

In view of the elementary nature of the book, we do not present a detailed bibliography on integral geometry and restrict ourselves to references to some publications mentioned in our exposition.

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