Preface

The title of this book, *Painlevé equations*, refers to a set of six nonlinear ordinary differential equations denoted traditionally by P_{I}, P_{II}, ..., P_{VI}. The first one

$$P_{\mathrm{I}}: \quad y'' = 6y^2 + t$$

appears to be the simplest among the six. Here t stands for the independent variable, $y = y(t)$ for the dependent variable (or the unknown function), and $' = d/dt$ for the derivative with respect to t. This is a *nonlinear* differential equation since its right-hand side is a quadratic function of y. As you go on to P_{II}, P_{III}, ..., the Painlevé equations take on a far more complicated appearance. I think you might be surprised if you are seeing P_{VI} for the first time! (See Table 0.1.)

The six Painlevé equations were discovered about a hundred years ago by a French mathematician named Paul Painlevé (1863–1933). Their solutions are now called the *Painlevé transcendents*. (Incidentally Painlevé was a politician too; you could see a photo of Painlevé celebrating Lindbergh, the first person to fly an airplane from New York to Paris.) You might ask "Could there be anything left unstudied about differential equations that have been known for a hundred years?" I would say, "Yes, they are *special*." In fact the history of the Painlevé equations reads like a detective story. But the Painlevé equations themselves are really a wonder. They still continue to give us fresh mysteries. You may suspect I am exaggerating, but this is my true impression. One reason I wrote this book is to tell you how impressed I am by the mysteries of the Painlevé equations.

The main theme of this book is the *symmetry* of the Painlevé equations. I will focus on P_{II} and P_{IV}, namely,

$$\begin{aligned} P_{\mathrm{II}}: &\qquad y'' = 2y^3 + ty + \alpha, \\ P_{\mathrm{IV}}: &\qquad y'' = \frac{1}{2y}(y')^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2 - \alpha)y + \frac{\beta}{y}, \end{aligned}$$

as examples, since they are easier to handle than the others. The symbols α and β appearing in these equations are parameters. If you are given a solution of a Painlevé equation with some parameters, you can use a special device, called a *Bäcklund transformation*, for creating a new solution with different parameters. "Symmetry" is a word used frequently to refer to such a mechanism to construct new solutions by transformations. I will begin by explaining what a Bäcklund transformation is and tell the story of the symmetry of the Painlevé equations on the basis of explicit examples. I hope you enjoy seeing how new solutions are generated, one after another, starting from simple solutions. I will also introduce the τ-functions (tau functions) and describe symmetry as a discrete system with respect to the parameters. After that I will formulate (in Chapter 5) the *Jacobi-Trudi formula*, a way to express explicitly all the solutions obtained by Bäcklund transformations.

Table 0.1. The Six Painlevé Equations

$$
\begin{aligned}
P_{\mathrm{I}}: &\quad y'' = 6y^2 + t \\
P_{\mathrm{II}}: &\quad y'' = 2y^3 + ty + \alpha \\
P_{\mathrm{III}}: &\quad y'' = \frac{1}{y}(y')^2 - \frac{1}{t}y' + \frac{1}{t}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y} \\
P_{\mathrm{IV}}: &\quad y'' = \frac{1}{2y}(y')^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2 - \alpha)y + \frac{\beta}{y} \\
P_{\mathrm{V}}: &\quad y'' = \left(\frac{1}{2y} + \frac{1}{y-1}\right)(y')^2 - \frac{1}{t}y' \\
&\qquad + \frac{(y-1)^2}{t^2}\left(\alpha y + \frac{\beta}{y}\right) + \frac{\gamma}{t}y + \delta\frac{y(y+1)}{y-1} \\
P_{\mathrm{VI}}: &\quad y'' = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)(y')^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)y' \\
&\qquad + \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right)
\end{aligned}
$$

In this list, $y = y(t)$ is the dependent variable and $' = d/dt$ stands for the derivative with respect to the independent variable t. The symbols $\alpha, \beta, \ldots$ are parameters.

The first five chapters are the main part of this book. The chapters that follow are an essay on *discrete systems of type A*, which generalize the symmetry structure of P_{II} and P_{IV}. There I give a proof of the Jacobi-Trudi formula in the general setting. (By an "essay" I mean an attempted explanation which some experts might think incomplete.)

I do not think you will find it difficult to read the main part of this book if you know the calculus and linear algebra learnt by first year undergraduates, and have a little knowledge of group theory (such as the definition of a group, and a group defined by generators and fundamental relations). I have chosen the subjects of this book so that you can directly approach the interesting aspects of Painlevé equations without any particular prerequisite. One nice thing about the Painlevé equations is that they are given in explicit forms and one can do computations with them. My hope is that you will try computations by hand (or by computer) while reading this book.

You might expect that a book on Painlevé equations should include such topics as how Painlevé discovered them, what their meaning is and what research has been done so far on them. I did not write much about such things, as they can be found in other books. One example of what I did not explain in this book is:

> Any rational ordinary differential equation of second order that has no movable branching points can be transformed into one of the Painlevé equations, unless it reduces to a simpler equation such as linear equations or differential equations for elliptic functions.

After considering several possibilities, I finally decided it would be very hard to maintain the intended style of this book if I were to include such topics. If you already have some more mathematical background, and want to learn more about Painlevé equations, including their historical aspects, I would recommend this excellent reference:

> Kazuo Okamoto, *Introduction to Painlevé Equations*, Sophia Kokyuroku in Mathematics, No. 19, 1985 (in Japanese)[1].

If you are interested, please browse Section 0 in the appendix (⇒ Appendix 0: Profile of Painlevé equations), where I give some remarks on the origin and the history of Painlevé equations (without detailed explanation of the terminology). In the appendix I have also collected some other topics, including things that are slightly removed from the main flow of this book, that I could not write in sufficient detail in the text, and that I thought to be more advanced. I hope the reader will find the short stories in the appendix helpful.

For several years I have been working with Yasuhiko Yamada, one of my colleagues at Kobe University, on Painlevé equations and related discrete integrable systems. In this book I tried to include extensively what has been made clear through my collaboration with him. The Jacobi-Trudi formula I introduce in Chapter 5, for example, is originally due to Yamada. This book thus contains new material, so in this sense is a *special course* on Painlevé equations rather than a standard textbook. Please keep this in mind as you read this book. I would like to express my thanks to Yasuhiko Yamada, to whom I owe uncountably many valuable suggestions, ever since the planning stages of this book. Without his cooperation, I cannot imagine that this book could have been completed. I am also grateful to the members of the mathematics department at Kobe University, especially to Kyoichi Takano, Kenji Iohara and Jun Nakagawa, who read the manuscript and kindly provided much advice for improving this book.

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[1] For English readers, I would suggest: K. Iwasaki, H. Kimura, S. Shimomura and M. Yoshida, *From Gauss to Painlevé — A Modern Theory of Special Functions*, Aspects of Mathematics E16, Vieweg, 1991.

Preface to the English Edition

This book was originally written in Japanese as a volume of the series of mathematical readings “Landscapes of Mathematics” (Asakura Shoten Publishers). My idea in writing this book was to provide an introduction to recent studies on the symmetry of Painlevé equations. I also tried to write the book in a plain language so that it can be read by a reader who has a mathematical background at the undergraduate level.

In order to fulfill this requirement, perhaps I should have added a section on the basic facts concerning groups defined by generators and relations. For those who are not familiar with abstract algebra, the following explanation might be sufficient as a starting point: A *group* is a collection of transformations in which one can compose transformations and take inverse transformations. When one wants to consider several *fundamental transformations* (called the generators) with prescribed *fundamental relations* among them, it is useful to consider the collection of all possible composed transformations among which all the relations are reducible to the fundamental ones. Such a group is the group defined by generators and relations.

As I already mentioned in the preface of the original edition, this book is not meant to be a standard textbook on Painlevé equations, but a special course on their symmetry aspects. It will be my pleasure if this book could help the reader to realize the richness of various mathematical structures appearing there, and in particular the remarkable role of affine Weyl groups in Painlevé equations.

I am grateful to the editors of the AMS for providing me with a chance to publish the English edition of my book. I also express my thanks to Professors Katusmi Nomizu, Wayne Rossman and Nicholas Witte for the many valuable comments they kindly offered during the preparation of the English edition.

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