Contents

Summary and Overview

The central theme of this book is the singular perturbation theory of differential equations, mainly the so-called WKB analysis. As Chapter 4 presents the analysis of the Painlevé transcendents whose final target is their connection formulae, the reader might wonder how it is related to the WKB analysis. But, as the reader will see, our discussion in Chapter 4 is based on the WKB analysis of a particular Schrödinger equation that underlies the Painlevé equation. In what follows, we give a summary of the theory of WKB analysis and the contents of this book, touching upon the historic background of the theory.

As explained in Chapter 2, WKB analysis (or the WKB method) is a method of obtaining a formal solution $\psi(x, \eta)$ of a (1-dimensional) Schrödinger equation

$$(1) \qquad \left(-\frac{d^2}{dx^2} + \eta^2 Q(x)\right)\psi(x, \eta) = 0$$

in the form

$$(2) \qquad \exp\left(\int_{x_0}^{x} S(x, \eta)dx\right),$$

where

$$(3) \qquad S(x, \eta) = S_{-1}(x)\eta + S_0(x) + S_1(x)\eta^{-1} + \cdots,$$

(4) $\qquad x_0$ is a properly chosen constant.

(Here $Q(x)$ is a holomorphic function or a rational function, and $\eta = 1/\hbar$, where $\hbar$ is the Planck constant. Hence, η is interpreted as a large parameter.) Such a formal solution is said to be a *WKB solution*. WKB is named for the three physicists Wentzel, Kramers, and Brillouin, who used this method efficiently for the study of quantum physics. As is usually the case with singular perturbations, this approach is a very natural one and the expansion of this sort had been used in analysis before them; as examples, we may count Jeffreys [**31**] and the Debye expansion for the Bessel function with a large order (cf. [**46**, p. 156] for example).* Actually the Debye formula, which may appear to be really a mysterious one, is understood to be a natural one if regarded as an example of the application of this method (cf. Chapter 2, Remark 2.3). Parenthetically, we note that we discuss in this book not a 'differential equation with a small parameter' but a 'differential equation with a large parameter (to be always denoted by η)', as we used $\eta = 1/\hbar$, not $\hbar$ itself, as a parameter in (1). This is a matter of convenience, but we call the reader's attention to this point in view of its importance; we use a large parameter mainly because the Borel transformation, which plays the central role in this book, can be more neatly described with the use of a large parameter, and the reason why we stick to the symbol η is just because the resulting notations are most well-balanced if the corresponding variable is denoted by y in the Borel transformation. (See Chapter 2, (2.20) for example.)

As will be explained in Chapter 2, Section 2.2, the formal solution $S(x, \eta)$ as given in (3) is uniquely determined recursively by

$$S_{-1}^2 = Q, \tag{5}$$

$$2S_{-1}S_j + \sum_{\substack{k+l=j-1 \\ k,l \geq 0}} S_k S_l + \frac{dS_{j-1}}{dx} = 0 \qquad (j \geq 0) \tag{6}$$

once the sign of $S_{-1} = \pm\sqrt{Q(x)}$ is fixed. From this construction, each S_j is holomorphic except at a zero point (which is called a *turning point* of (1)) and a singular point of $Q(x)$. Thus, the algebraic structure is clear. However, unfortunately, $S(x, \eta)$ almost always diverges as a series in η^{-1} (Chapter 2, Sections 2.2 and 2.3), reflecting its singular perturbative character. And, as is always the case with divergent series, disputes over the legitimacy and the scope of applicability of WKB analysis continued until quite recently; such foggy conditions were cleared up only in the 1980's by the key-word 'Borel sum' (Chapter 1). To be more concrete, by considering the Borel

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Note added in proof* (August, 2005): Concerning this point Professor J. J. Duistermaat has kindly informed one of the authors (T.K.) that WKB expansions follow directly from the techniques of Liouville [44**] and that we should not forget the original form of the Sturm-Liouville theory. We share the same viewpoint of the Sturm-Liouville theory with Professor Duistermaat, and we are most grateful to him for his kind and informative letter dated September 4, 2000.

sum of WKB solutions, Voros [**65**] established in 1983 the connection formulae for WKB solutions (cf. Chapter 2, Section 2.3) when all the zero points of $Q(x)$ are simple, and, in 1985, Silverstone [**57**] gave a clear-cut argument to show that the doubts over the legitimacy of the WKB analysis are due to the arguments applied—without any logical grounds—to the points where the Borel sum of a WKB solution is not well defined. We note that, behind these epoch-making papers, there had been, on the mathematics side, the suggestion of Dingle [**20**] to the effect that correct connection formulae for WKB solutions could be obtained only through the information on all terms even though they are divergent, and, on the physics side, several affirmative results on the Borel summability of perturbative expansions that followed the remarkable paper of Bender-Wu [**14**]. (Cf. e.g. Magnen-Sénéor [**45**], 't Hooft [**64**] and Eckmann-Epstein [**24**].) Further, just around that time Ecalle ([**21**], [**22**], etc.) was developing a new analysis 'resurgent theory', which is also based on the Borel sum, and located at the crossing point of these two trends are the works of the Nice group led by Pham (Pham [**51**], Candelpergher-Nosmas-Pham [**16**], Delabaere-Dillinger-Pham [**18**], [**19**], etc.). This kind of WKB analysis based on Borel sums has been recently called the exact WKB analysis. In this book, in most cases, we simply say 'the WKB analysis' for the exact WKB analysis. (Ignoring whether a 'proof' is given or not (actually the argument on the Borel summability in Chapter 4 is not yet perfect; see Future Directions and Problems), as an idea, we are always considering the exact WKB analysis.)

The discussion given so far might have given the reader an impression that the exact WKB analysis was developed just to avoid divergence problems. However, the consideration of the Borel sum of a WKB solution has much more positive merit; as will be explained in Chapter 1, the notion of a Borel sum is based on the analytic continuation of the *Borel transform* of a divergent series, giving the exact description of the divergent series. The above fact is the reason why the exact WKB analysis is effective for the treatment of exponentially small terms in eigenvalue problems, but the argument of Voros in [**65**] is effective for general problems of differential equations in the large, beyond the framework of eigenvalue problems ([**2**], [**54**]). As one of the most outstanding examples, we will show the fact (see Chapter 3, Section 3.1) that the monodromy group of a second order Fuchsian type differential equation (in a generic situation) can be described in terms of 'the contour integral of the logarithmic derivative of the

WKB solution' (precisely speaking, of its odd part) (see Chapter 2, Section 2.1). The reader will get an impression 'Yes, indeed exact'.

Now that the WKB analysis is shown to be effective for describing the monodromy group, the following question can be naturally raised: How is the WKB analysis related to the monodromy preserving deformation? (See Jimbo [**32**].) (Note that 'naturally' does not mean 'trivially'. As we mentioned in the Preface, it was Professor Michio Jimbo who led us to this question.) We initially considered this problem to be easy to answer. However, when we began our investigation by introducing a large parameter η (Chapter 4, Section 4.1), to our surprise, the monodromy preserving deformation is always associated with a double turning point (Chapter 4, Section 4.3). This kind of an inevitable degeneration often indicates something interesting behind it; in fact, we found an analysis centered around this 'double turning point'. To be more explicit, we first construct a formal solution (with the double turning point as the principal part) of the Painlevé equation relevant to the monodromy preserving deformation in question. Then we look for the connection formula for general Painlevé transcendents by using the following properties (7) and (8). (For the history of the Painlevé transcendent, see [**32**] by Professor Jimbo, who has made a substantial contribution to it. We like to note, at least, that it is an interesting example to be seriously considered when we discuss the problem: what is 'useful mathematics'?, and that about one hundred years after the work of Painlevé, the Painlevé transcendent (a special function of the twentieth century) is still an attractive subject to study, which has not yet been fully understood.)

(7) For Painlevé I, the analytic continuation of the Borel sum of the formal solution can be explicitly described (Chapter 4, Section 4.5).

(8) With an appropriate correspondence of the parameters, their formal solutions are locally 'equivalent' (see Chapter 4, Section 4.6 for the precise meaning).

Although we do not describe the details of our discussion here (see Chapter 4 for them), we should emphasize the following fact.

"The 'transformation' to be used to show the 'equivalence' asserted in (8) should be, logically speaking, found by studying the Painlevé equation only; but, in our actual construction of the 'transformation', the transformation of the Schrödinger equation (SL_J) that underlies the Painlevé equation (P_J) appears naturally."

Here we should recall the fact that the Painlevé equation was originally found by Painlevé (and his student Gambier) while looking for a second order differential equation whose solutions do not admit movable branch points. This viewpoint is completely different from what is employed in this book, i.e., '(P_J) is a condition that lets (SL_J) be deformed isomonodromically'. (See R. Fuchs [**28**] for $(P_{\rm VI})$, and Okamoto [**48**] for other (P_J)'s. See also Jimbo-Miwa-Ueno [**33**] and Jimbo [**32**].) Even the formulation should be difficult, if we were to study connection formulae for solutions of an equation which admits solutions with movable branch points, i.e., branch points that depend on parameters contained in the solution (not the equation) [note that the singular points of solutions of linear differential equations (cf. Chapters 2 and 3) are confined to the singular points of the equation and that such movable branch points do not appear for linear equations]. Honestly speaking, we 'naturally' arrived at the connection formulae for the Painlevé transcendents, without being seriously aware of the above characterization (the so-called Painlevé property) of the Painlevé transcendents, at least at the first stage of our study. In retrospect, we ourselves are really impressed by the miraculous harmony that the Painlevé transcendents enjoy. We are, however, still far away from the level of 'applicable mathematics'; e.g., we do not know how our constructed formal solution corresponds to the true solution. We hope that some of the readers will join us in this quest. Shall we make a collection of formulas where the Painlevé transcendent is a twentieth century special function?

CHAPTER 1

Borel Resummation

We will give a concise description of Borel resummation, which will be the foundation of WKB analysis. Various examples should serve as an introduction to WKB analysis. After reviewing the definition of a Borel resummation and its basic properties, as a concrete example, we will study a formal solution (at infinity) of the Weber equation in terms of Borel resummation. The explicit computation of a formal solution for the Weber equation (especially on pages 7–9) has a strong connection to the determination of the Stokes multipliers (at the unique irregular singularity at infinity) of the Weber equation. This computation is a prototype explaining the mechanism of 'Stokes phenomenon' (which is described by the connection formula) for the WKB solution in Chapter 2.

As a typical method to seek for a solution of a differential equation, we often express an unknown function as a power series of independent variables (or parameters). However, a power series that is a formal solution at an irregular singularity, or for an equation for singular perturbation, often does not converge. In this chapter, we will describe the method of a *Borel resummation* through examples, keeping in mind its relevance to differential equations. A Borel resummation provides an analytic meaning to such a divergent series. (For a systematic treatment of a Borel resummation and its history, see Hardy [**29**], Ezawa [**25**], Balser [**12**], etc.) We begin with the definitions of the Borel transformation and the Borel sum.

Definition 1.1. Let α be a real number satisfying $\alpha \notin \{0, -1, -2, \dots\}$. For a formal power series with respect to z^{-1} of the form

$$f = \exp(\zeta_0 z) \sum_{n=0}^{\infty} f_n z^{-\alpha-n}, \qquad z > 0, \tag{1.1}$$

where ζ_0 and f_n are constants, the *Borel transform* $f_B(\zeta)$ is defined by

$$f_B(\zeta) = \sum_{n=0}^{\infty} \frac{f_n}{\Gamma(\alpha + n)} (\zeta + \zeta_0)^{\alpha+n-1}, \tag{1.2}$$

where Γ stands for the gamma function. Furthermore, when the Laplace integral

$$\int_{-\zeta_0}^{\infty} \exp(-z\zeta) f_B(\zeta)\, d\zeta \tag{1.3}$$

exists, the integral (1.3) is said to be the *Borel sum* of the formal power series f.

REMARK 1.2. The integration path in (1.3) is extended from $-\zeta_0$ to a point at infinity parallel to the positive real axis. This choice of the integration path corresponds to the positiveness of z. If z is a large complex number moving on the line $\mathbb{R}_+ \exp(i\theta)$, i.e., the half positive real axis rotated by angle θ, then the integration path in (1.3) should also be rotated in the complex plane. (See the explanation following Example 1.4.) However, in this book, the variable z appearing in Borel sums is always positive (otherwise stated, a Borel sum is defined by the Laplace integral (1.3) along the integration path).

As is well known, for the Heaviside function $Y(\zeta)$ (i.e., $Y(\zeta) \equiv 1$ for $\zeta > 0$ and $Y(\zeta) \equiv 0$ for $\zeta < 0$) $\zeta^{\alpha+n-1} Y(\zeta)/\Gamma(\alpha+n)$ has $z^{-(\alpha+n)}$ as its Laplace transform. Therefore, the Borel transformation in Definition 1.1 can be considered as the formal generalization to a series as in (1.1) of the inverse Laplace transformation. The Borel sum given as the integral (1.3) is precisely the Laplace transform. Therefore, it is natural that the Borel resummation should be expected to be a 'natural' resummation for a divergent series. When one considers a convergent series for f in (1.1), its Borel transform $f_B(\zeta)$ (more precisely, $(\zeta+\zeta_0)^{1-\alpha} f_B(\zeta)$ should be considered instead of $f_B(\zeta)$) becomes an entire function of an exponential type (namely, an analytic function bounded by $C \exp A|\zeta|$ from above over $\mathbb{C}$, where C and A are positive constants). The Laplace integral (1.3) has a meaning for a sufficiently large z and equals the convergent series f (the regularity of Borel resummation). Furthermore, even for a non-convergent series f, if f is Borel summable in the sense of the following Definition 1.3, the original f can be recovered as an asymptotic expansion from the Borel sum. (See, e.g., Ezawa [**25**], §4.4 and Theorem 4.4, for a proof.)

DEFINITION 1.3. If a formal power series f of the form (1.1) satisfies the following three conditions (i), (ii), (iii), then f is said to be *Borel summable*.

(i) $(\zeta+\zeta_0)^{1-\alpha} f_B(\zeta) = \sum_{n=0}^{\infty} \frac{f_n}{\Gamma(\alpha+n)} (\zeta+\zeta_0)^n$ converges at $\zeta = -\zeta_0$.
(ii) $f_B(\zeta)$ can be analytically continued to a domain containing $\{\zeta \in \mathbb{C}; \Im(\zeta+\zeta_0) = 0 \text{ and } \Re(\zeta+\zeta_0) > 0\}$ in the ζ-plane.
(iii) For a sufficiently large z, $\int_{-\zeta_0}^{\infty} \exp(-z\zeta) f_B(\zeta)\, d\zeta$ has a finite fixed value.

In particular, when (i) is satisfied, namely, where positive constant numbers A and C exist to satisfy

$$|f_n| \le AC^n n! \tag{1.4}$$

for any natural number n, then the Borel transform $f_B(\zeta)$ of f becomes an analytic function of ζ. When a formal power series f satisfies (1.4), f is said to be *Borel transformable*.

Thus, Borel resummation is a series resummation. The definition of Borel resummation also indicates that Borel resummation has a strong (or stronger than expected) connection to differential equation theory via Laplace analysis. (The WKB analysis for the Schrödinger equation in Chapter 2 is an example of this connection.) As an introduction to Chapter 2, we will give explicit and simple examples showing the connection between Borel resummations and differential equations.

EXAMPLE 1.4. Consider the following formal power series:

$$f = \sum_{n=0}^{\infty} (-1)^n n!\, z^{-n-1}. \tag{1.5}$$

This f is a formal solution of the ordinary differential equation

$$\left(-\frac{d}{dz} + 1\right) \psi(z) = \frac{1}{z} \tag{1.6}$$

at the (irregular) singular point $z = \infty$. By the definition, the Borel transform $f_B(\zeta)$ becomes

$$f_B(\zeta) = \sum_{n=0}^{\infty} (-1)^n \zeta^n = \frac{1}{1+\zeta}. \tag{1.7}$$

Note that $f_B(\zeta)$ has the singularity $\zeta = -1$ reflecting the divergence of f. Then the Borel sum

$$\int_0^\infty \exp(-z\zeta) \frac{1}{1+\zeta}\, d\zeta \tag{1.8}$$

is a true solution (an analytic solution) of the differential equation (1.6) (in an open angular domain containing the positive real axis with vertex $z = \infty$).

In Example 1.4, z is restricted to being positive real. However, in order to relate f with the differential equation (1.6), the restriction $z > 0$ seems somewhat unnatural. Hence we will consider changing the direction of the variable z. As in Definition 1.1, now assume z satisfies $z = r\exp(i\theta)$ $(\theta \in \mathbb{R}, r > 0)$. Then regard

$$f = \exp(\zeta_0 z) \sum_{n=0}^{\infty} f_n z^{-\alpha-n} = \exp(\zeta_0 r e^{i\theta}) \sum_{n=0}^{\infty} f_n r^{-\alpha-n} e^{-i\theta(\alpha+n)}$$

as a formal power series of r. Then its Borel transform is given by

$$\sum_{n=0}^{\infty} \frac{f_n e^{-i\theta(\alpha+n)}}{\Gamma(\alpha+n)} (\rho + \zeta_0 e^{i\theta})^{\alpha+n-1} = e^{-i\theta} f_B(\rho e^{-i\theta}).$$

Therefore, the Borel sum becomes

$$\int_{-\zeta_0 e^{i\theta}}^{\infty} \exp(-r\rho) f_B(\rho e^{-i\theta}) e^{-i\theta} d\rho = \int_{-\zeta_0}^{e^{-i\theta}\infty} \exp(-z\zeta) f_B(\zeta)\, d\zeta,$$

where $e^{-i\theta}\infty$ indicates the integration from $-\zeta_0$ to infinity parallel to the segment $\arg\zeta = -\theta$. Namely, the shift of the argument by θ for the Borel resummation corresponds to rotating the integration path by $-\theta$ in the Laplace integral (1.3) determining the Borel sum.

In Example 1.4, the integral (1.8) giving the Borel sum of the formal solution f was finite and determined for $z > 0$. This is because the singular point $\zeta = -1$ of the Borel transform $f_B(\zeta) = 1/(1+\zeta)$ does not meet the integration path. When the argument of z is changed, the direction of the integration path of (1.8) is changed. For example, for $\arg z = \pm\pi$, (1.8) is the integral along the negative real axis. Then we are faced with the singularity $\zeta = -1$ on the integration path. As will be clarified by the discussion following Example 1.6, this phenomenon is closely related with the *Stokes phenomenon* at an irregular singular point ($z = \infty$ in Example 1.6) for a differential equation. Here the Stokes phenomenon means a phenomenon that, in a neighborhood of an irregular singular point, an analytic solution having the same formal solution as an asymptotic series can differ depending upon the direction of an approach to the singular point. (For irregular singular points and Stokes phenomenon, see, for example, Okubo-Kohno [**50**] or Takano [**58**].)

To see the relationship between a singular point of the Borel transform $f_B(\zeta)$ and the Stokes phenomenon in detail, we will explicitly compute the Borel sum of a formal solution of a typical example of a second order homogeneous equation, i.e., the Weber equation.

EXAMPLE 1.5. Let $\lambda \notin \{0, 1, 2, \dots\}$. Consider the following formal series:

$$\exp\left(-\frac{z^2}{4}\right) z^{\lambda} \sum_{n=0}^{\infty} (-1)^n \frac{\lambda(\lambda-1)\cdots(\lambda-2n+1)}{n!\, 2^n z^{2n}}. \tag{1.9}$$

This formal series is a formal solution of the Weber differential equation

$$\frac{d^2\psi}{dz^2} + \left(\lambda + \frac{1}{2} - \frac{z^2}{4}\right)\psi = 0 \tag{1.10}$$

at the irregular singular point $z = \infty$.

Let

$$f = z^{\lambda} \sum_{n=0}^{\infty} (-1)^n \frac{\lambda(\lambda-1)\cdots(\lambda-2n+1)}{n!\, 2^n z^{2n}}, \tag{1.11}$$

and let us compute the Borel sum of f: The Borel transform $f_B(\zeta)$ of f is given by

$$\begin{aligned} f_B(\zeta) &= \sum_{n=0}^{\infty} (-1)^n \frac{\lambda(\lambda-1)\cdots(\lambda-2n+1)}{n!\, 2^n \Gamma(-\lambda+2n)} \zeta^{2n-\lambda-1} \\ &= \sum_n \frac{(-1)^n}{n!\, 2^n \Gamma(-\lambda)} \zeta^{2n-\lambda-1} \\ &= \frac{\zeta^{-\lambda-1}}{\Gamma(-\lambda)} \exp\left(-\frac{\zeta^2}{2}\right). \end{aligned} \tag{1.12}$$

Therefore, the Borel sum of the formal solution (1.9) becomes

$$\exp\left(-\frac{z^2}{4}\right) \frac{1}{\Gamma(-\lambda)} \int_0^{\infty} \exp\left(-z\zeta - \frac{\zeta^2}{2}\right) \zeta^{-\lambda-1}\, d\zeta. \tag{1.13}$$

(Note that for $\Re\lambda < 0$, the above integral is finite and determined in the usual sense.) The Borel sum (1.13) is precisely an integral representation of the Weber function (parabolic cylinder function)

$D_\lambda(z)$ which is a holomorphic solution of the Weber differential equation (1.10). It is a well-known fact that the asymptotic expansion of $D_\lambda(z)$ is given by the formal series (1.9) (see [**46**, §19]).

In Example 1.5, the Borel transform $f_B(\zeta)$ of f is an entire function (except the part of $\zeta^{-\lambda-1}$). However, as can be observed from the Borel sum (1.13), $\arg z$ cannot be changed beyond $\pm\frac{\pi}{4}$ (since the integral would diverge because of the factor $\exp(-\frac{\zeta^2}{2})$).

Example 1.6. We will consider the same formal series as (1.9). This time we let $z^2 = y$. Then we apply the Borel resummation to

$$g = \exp\left(-\frac{y}{4}\right) y^{\lambda/2} \sum_{n=0}^{\infty} (-1)^n \frac{\lambda(\lambda-1)\cdots(\lambda-2n+1)}{n!\,2^n y^n}. \tag{1.14}$$

We will begin by computing the Borel transform $g_B(\eta)$:

(1.15)

$$\begin{aligned}
g_B(\eta) &= \sum_n (-1)^n \frac{\lambda(\lambda-1)\cdots(\lambda-2n+1)}{n!\,2^n\Gamma(-\frac{\lambda}{2}+n)} \left(\eta - \frac{1}{4}\right)^{-\lambda/2+n-1} \\
&= \sum_n \frac{(-1)^n\lambda(\lambda-1)\cdots(\lambda-2n+1)}{n!\,2^n(-\frac{\lambda}{2}+n-1)\cdots(-\frac{\lambda}{2})\Gamma(-\frac{\lambda}{2})} \left(\eta - \frac{1}{4}\right)^{-\lambda/2+n-1} \\
&= \frac{(\eta-\frac{1}{4})^{-\lambda/2-1}}{\Gamma(-\lambda/2)} \sum_n \frac{(\lambda-1)(\lambda-3)\cdots(\lambda-2n+1)}{n!} \left(\eta - \frac{1}{4}\right)^n \\
&= \frac{(\eta-\frac{1}{4})^{-\lambda/2-1}}{\Gamma(-\lambda/2)} \sum_n \frac{\frac{\lambda-1}{2}\cdots(\frac{\lambda-1}{2}-n+1)}{n!} \left(2\eta - \frac{1}{2}\right)^n \\
&= \frac{(\eta-\frac{1}{4})^{-\lambda/2-1}}{\Gamma(-\lambda/2)} \left(1 + 2\eta - \frac{1}{2}\right)^{(\lambda-1)/2} \\
&= \frac{2^{(\lambda-1)/2}}{\Gamma(-\lambda/2)} \left(\eta - \frac{1}{4}\right)^{-(\lambda+2)/2} \left(\eta + \frac{1}{4}\right)^{(\lambda-1)/2}.
\end{aligned}$$

Therefore (by regarding $z^2 = y$ as a new variable), the Borel sum is

(1.16)

$$\begin{aligned}
&\frac{2^{(\lambda-1)/2}}{\Gamma(-\lambda/2)} \int_{1/4}^{\infty} \exp(-z^2\eta) \left(\eta - \frac{1}{4}\right)^{-(\lambda+2)/2} \left(\eta + \frac{1}{4}\right)^{(\lambda-1)/2} d\eta \\
&= \frac{2^{(\lambda-1)/2}}{\Gamma(-\lambda/2)} \int_0^{\infty} \exp\left(-z^2\left(\eta + \frac{1}{4}\right)\right) \eta^{-(\lambda+2)/2} \left(\eta + \frac{1}{2}\right)^{(\lambda-1)/2} d\eta.
\end{aligned}$$

This gives another integral representation of the Weber function $D_\lambda(z)$ differing from (1.13).

In contrast to Example 1.5, in Example 1.6 the Borel transform $g_B(\eta)$ of g also has the singular point at $\eta = -\frac{1}{4}$ (in addition to the 'base point' $\eta = \frac{1}{4}$, whose existence is a part of the definition of Borel transformation). This 'new singular point' $\eta = -\frac{1}{4}$ is connected to the Stokes phenomenon at $z = \infty$ of the differential equation (1.10), as we see below.

The Weber equation (1.10) has a (unique) irregular singular point at $z = \infty$, at which linearly independent formal solutions are given by

$$\exp\left(\frac{z^2}{4}\right) z^{-\lambda-1} \sum_{n=0}^{\infty} \frac{(\lambda+1)(\lambda+2)\cdots(\lambda+2n)}{n!\, 2^n z^{2n}} \tag{1.17}$$

and (1.9). For those formal solutions, we will consider the Borel sum in the sense of Example 1.6. As observed in Example 1.6, for $z > 0$, the Borel sum (1.16) coincides with the Weber function $D_\lambda(z)$. Furthermore, the Borel sum can be analytically continued to $\arg y = \pm\pi$, i.e., $\arg z = \pm\frac{\pi}{2}$. Similarly, consider the Borel sum of (1.17) on $i\mathbb{R}_+$. There the Borel sum coincides with $\exp(-i\pi(\lambda+1)/2)D_{-\lambda-1}(-iz)$ and can be analytically continued to $\arg z \in (0, \pi)$. Furthermore, for $z < 0$, the Borel sum of (1.9) coincides with $\exp(i\pi\lambda)D_\lambda(-z)$ and can be analytically continued to $\arg z \in (\pi/2, 3\pi/2)$. Namely, in the angular domains

$$V_1 = \{z \in \mathbb{C}; 0 < \arg z < \pi/2\},$$
$$V_2 = \{z \in \mathbb{C}; \pi/2 < \arg z < \pi\}$$

(around $z = \infty$), the Borel sums of the linearly independent formal solutions (1.9) and (1.17) are determined, coinciding with

$$\text{in } V_1,\ D_\lambda(z),\ \exp(-i\pi(\lambda+1)/2)D_{-\lambda-1}(-iz), \tag{1.18}$$

$$\text{in } V_2,\ \exp(i\pi\lambda)D_\lambda(-z),\ \exp(-i\pi(\lambda+1)/2)D_{-\lambda-1}(-iz). \tag{1.19}$$

Notice that if the Borel sum $D_\lambda(z)$ of (1.9) is to be analytically continued beyond $\arg z = \frac{\pi}{2}$, then the 'new singular point' $\eta = -\frac{1}{4}$ of the Borel transform $g_B(\eta)$ would be on the integration path of the Borel sum. Namely, the integration path needs to be altered to analytically continue the Borel sum by avoiding this situation. Then the analytic continuation (for example, for $\arg z = \frac{\pi}{2} + \varepsilon$, $0 < \varepsilon \ll 1$) can be

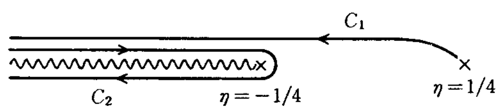

FIGURE 1.1. The integration paths C_1, C_2 giving the analytic continuation of the Borel sum (1.16) (where $\eta = \pm 1/4$ are singular points of $g_B(\eta)$, and the wavy line is the cut specifying the branch of $g_B(\eta)$).

realized by the sum of the following two integrals:

$$\int_{C_1} \exp(-z^2\eta) g_B(\eta)\, d\eta + \int_{C_2} \exp(-z^2\eta) g_B(\eta)\, d\eta. \tag{1.20}$$

See Figure 1.1 for the integration paths C_j.

The first term of (1.20) equals the Borel sum $\exp(i\pi\lambda)D_\lambda(-z)$ (or rather its analytic continuation) of (1.9) on V_2. On the other hand, the second term of (1.20) can be rewritten as follows:

$$\begin{aligned}
&\int_{C_2} \exp(-z^2\eta) g_B(\eta)\, d\eta \\
&\quad = \frac{2^{(\lambda-1)/2}}{\Gamma(-\lambda/2)} e^{i\pi(\lambda+2)/2} (e^{i\pi(\lambda-1)/2} - e^{-i\pi(\lambda-1)/2}) \\
&\qquad \times \int_0^\infty \exp\left(z^2\left(t+\frac{1}{4}\right)\right)\left(t+\frac{1}{2}\right)^{-(\lambda+2)/2} t^{(\lambda-1)/2}\, dt \\
&\quad = \frac{2^{(\lambda+1)/2}}{\Gamma(-\lambda/2)} e^{i\pi(\lambda+1)/2} \sin\frac{\pi(1-\lambda)}{2} \\
&\qquad \times \int_0^\infty \exp\left(z^2\left(t+\frac{1}{4}\right)\right)\left(t+\frac{1}{2}\right)^{-(\lambda+2)/2} t^{(\lambda-1)/2}\, dt.
\end{aligned}$$

By using the following formulas on the gamma function Γ ([**46**, p. 1])

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}, \tag{1.21}$$

$$\Gamma(2x) = \frac{2^{2x}}{2\sqrt{\pi}} \Gamma(x)\Gamma\left(x+\frac{1}{2}\right), \tag{1.22}$$

the right-hand side of the above equation becomes

$$\frac{\sqrt{2\pi}e^{i\pi(\lambda+1)/2}2^{-(\lambda+2)/2}}{\Gamma(-\lambda)\Gamma((\lambda+1)/2)} \times \int_0^\infty \exp\left(z^2\left(t+\frac{1}{4}\right)\right)\left(t+\frac{1}{2}\right)^{-(\lambda+2)/2} t^{(\lambda-1)/2}\,dt.$$

Therefore, the second term of (1.20) equals

$$\frac{\sqrt{2\pi}e^{i\pi(\lambda+1)/2}}{\Gamma(-\lambda)} D_{-\lambda-1}(-iz).$$

Consequently, we obtain the classical connection formula of the Weber function

$$D_\lambda(z) = \exp(i\pi\lambda)D_\lambda(-z) + \frac{\sqrt{2\pi}e^{i\pi(\lambda+1)}}{\Gamma(-\lambda)}\exp\left(-\frac{i\pi(\lambda+1)}{2}\right)D_{-\lambda-1}(-iz). \tag{1.23}$$

Note that the relations (1.18) and (1.19) between the formal solution and its Borel sum are valid for larger angular domains in the sense of asymptotic expansions ([**46**, p. 77]). Hence, the formula (1.23) expresses a relation between holomorphic solutions having the same formal solution (1.9) as asymptotic series in different domains, i.e., the Stokes phenomenon. Traditionally the 'Stokes multiplier of the Weber differential equation (1.10)' means the non-trivial off-diagonal component (i.e., $\sqrt{2\pi}\exp(i\pi(\lambda+1))/\Gamma(-\lambda)$) of the matrix

$$\begin{pmatrix} 1 & 0 \\ \sqrt{2\pi}e^{i\pi(\lambda+1)}/\Gamma(-\lambda) & 1 \end{pmatrix}$$

that relates two fundamental systems of solutions

$$(D_\lambda(z), \exp(-i\pi(\lambda+1)/2)D_{-\lambda}(-iz))$$

and

$$(\exp(i\pi\lambda)D_\lambda(-z), \exp(-i\pi(\lambda+1)/2)D_{-\lambda-1}(-iz))$$

on the domain $\{z; \pi/4 < \arg z < 3\pi/4\}$, i.e., the common domain of the validity of their asymptotic expansions.

As seen from the explicit computation of the above example, the 'new singular point' of the Borel transform of the formal solution has a close connection to the Stokes phenomenon at an irregular singular point of the differential equation. Namely, the divergence of the formal solution lets us anticipate the existence of the 'new singular point', besides the base point, of the Borel transform, and this 'new singular point' is an obstacle to the analytic continuation of the Borel sum of the formal solution. The integration path in the Borel sum needs to be altered to perform the analytic continuation. As its consequence the Borel sum of the formal solution acquires a contour integral of the Borel transform around the 'new singular point'. This induces the traditionally well-known Stokes phenomenon. If we can describe the singular part of the Borel transform of the formal solution at its 'new singular point' we can then find the Stokes multiplier for the Stokes phenomenon.

It is noteworthy that the change of variables $y = x^2$ was effective in Example 1.6. In fact, we observed (i) the Borel transform including the factor $\exp(-y/4)$ can be considered, and (ii) the Borel transform $g_B(\eta)$ has the 'new singular point', as anticipated, and also: for $|\eta| \to \infty$, the growth becomes extremely tamed. In particular, because of (ii), the integration path can be changed satisfying $\Re z^2\eta > 0$ as z varies in the Laplace integral (1.16) determining the Borel sum of the formal solution, and the 'new singular point', which, at first glance, might appear an obstacle to the deformation of the integration path, has really given birth to the connection formula (1.23). As is evident from these observations, not only the singularities originating immediately from the formal solution (as is observed in (i)) but also the singularities (including its growth order) of its Borel transform should be in harmony (as is observed in (ii)) if we want to successfully perform the analytic continuation of the Borel transform of the formal solution. (The Stokes multiplier of the Weber differential equation possibly could not have been computed if the Borel sum in the sense of Example 1.5 were considered only.)

In Chapter 2 we study the (one-dimensional) Schrödinger equation from the viewpoint of the Borel summation, and the reader will find that the same mechanism as that described here works nicely there. The miraculous harmony indicates the large parameter (originating from the Planck constant) is built in the Schrödinger equation in a fantastically 'natural' manner.

Summary

1.1 The Borel sum of a formal power series is defined as the Laplace integral of its Borel transform.

1.2 Borel resummation assigns the Borel sum to a formal series. It is one of the most natural resummation methods—it gives us an analytic solution when adapted to a formal solution of a differential equation.

1.3 When a formal power series is a divergent series, the Borel transform in general has a 'new singular point'. Because of this 'new singular point', the Borel sum of a divergent formal solution of a differential equation has a meaning only in an angular domain.

1.4 The classical Stokes phenomenon occurs where the integration path of the Laplace integral meets a 'new singular point'. In particular, the singular part at the 'new singular point' of the Borel transform of the formal solution is closely related to the Stokes multiplier.