Preface

About functional analysis. There is a common opinion that algebra studies sets endowed with operations, such as, say, addition, multiplication, taking the inverse or symmetric element. On the other hand, topology studies sets where continuous passing from some elements to others makes sense, and in particular, where convergent sequences are defined. With the same oversimplification, we can say that functional analysis is the study of sets with synthetic (i.e., composite) structure, partly algebraic and partly topological, and the two parts are related to each other by certain natural rules.

In the original form of classical functional analysis of the 1920s the algebraic structure is the linear space structure, while the convergence of vectors is defined (and made compatible with algebraic operations) via a given norm. However, it was immediately discovered that for the main part the really deep results require the additional assumption of completeness of the normed space under consideration. This led to the notion of Banach space. But the greatest progress in classical functional analysis was achieved in the theory of Hilbert spaces, which are defined as Banach spaces with norm given by an inner product. Here "complete success" was achieved in a number of fundamental directions. Notably, we now know everything about the nature of Hilbert spaces (the Riesz–Fisher theorem) and about the most important classes of maps of these spaces (the Hilbert spectral theorem, the Schmidt theorem).

(The classical functional analysis of normed, Banach, and Hilbert spaces constitutes the main part of our book.)

As time passed, new problems required enrichment of the initial structures. First of all, it was realized that to study many important function spaces we cannot restrict ourselves to just one norm. The natural convergences in such spaces can be described only with the help of several norms, or, more precisely, several more general "prenorms". Thus polynormed spaces appeared, and we use them now in the theory of generalized functions, in complex analysis, and in differential geometry. They turned out to be useful in the classical functional analysis as well, supplying it with new types of convergence.

(A special chapter in our book is devoted to polynormed spaces and to some of their applications.)

Another important observation was made. A deep and beautiful theory can be built if, instead of enriching the topological structure of our spaces, we enrich their algebraic structure by adding a new operation, namely, multiplication of vectors. The theory of Banach algebras was constructed in this way. Its most significant results are two fundamental "realization theorems". The first one (the Gelfand theorem) asserts that, up to some explicitly measured imprecision, every commutative Banach algebra is the algebra of continuous functions on some topological space, with pointwise operations. The second theorem (the Gelfand–Naimark theorem, which later became indispensable in modern quantum physics) states that every Banach algebra endowed with an involution (i.e., a kind of natural symmetry) which is consistent with the norm, is just the algebra of operators in a Hilbert space.

(In our book we present only the elements of the theory of Banach algebras needed for a discussion of spectra. Both theorems are given without proof, but we believe that every student should know their statements.)

Finally, the last 20 years of the 20th century were the time of birth and rapid development of a new branch of our science—quantum functional analysis, or the theory of operator spaces. In this theory linear spaces are endowed not with a norm, but with a more complicated structure, sometimes called "quantum norm". Namely, every space of matrices (of arbitrary size) with entries in a given space has its own norm, and all these norms agree with each other in a certain reasonable way. It was realized that many difficult questions in analysis become clear and transparent if we pass from the given space to its "quantization". Some important problems, which for many years resisted solution by classical methods, were solved in this way (see [**1**]). At the same time, quantum functional analysis, in addition to clarifying many questions of classical analysis, has its own adornments in the form of deep and brilliant theorems that have no classical analogues.

(In this book we did not dare to plunge into quantum functional analysis. We only tried to satisfy possible curiosity of the reader and give (in small print) two main definitions: of a quantum space and of a completely bounded operator. One of the sections of our book is devoted to these definitions, with preliminaries and discussion. The corresponding text plays the role of an advertisement and is not necessary even for advanced students.)

We have mentioned only some parts of functional analysis. A more complete discussion would have touched upon its other branches. For instance, throughout its history, functional analysis developed in close connection with harmonic analysis. The latter, roughly speaking, studies "shifts in function spaces", and can be regarded as an integral part of functional analysis. (Elements of harmonic analysis are presented in our book in the chapter devoted to the Fourier transform.) The theory of ordered spaces and the theory of topological algebras took shape and became separate areas of functional analysis. Further, we emphasize that the classical functional analysis—geometry of Banach spaces—has by no means come to a standstill. So many times "Banach geometry" was predicted to retire, but it continues to astonish us with new, deep, and unexpected results. (Some latest achievements, such as Gowers' theorem, are mentioned in the book.)

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Of course, we would like the reader of the book to get the impression that functional analysis is a beautiful science, rich in content. But, apparently, the intrinsic value is not sufficient for the full development and long life of a mathematical discipline. Every mathematical science that loses connections with the rest of mathematics, runs the danger of becoming an "art for art's sake", as von Neumann wrote in [**2**]. As a result, we can see that in some areas, smaller and smaller questions are studied under stronger and stronger magnification, whereas in other areas (or, sometimes, in the same ones) feeble "general theories" arise with an evidently insufficient number of substantial examples.

But we can reassure the reader: these problems do not threaten functional analysis now. There are many connections with the surrounding areas of mathematics and with other sciences, and these connections are being renewed all the time. Indeed, what is mathematics? The accepted answer is now: the science of patterns. But physics is teeming with patterns from functional analysis: from the very early ones, which appeared in the calculus of variations, to the ultramodern, based on recent achievements in the theory of operator algebras (see for example [**3**]). (The most famous pattern belongs to von Neumann, who dared to attack quantum mechanics as a whole.) And within mathematics, functional analysis allows us to consider, from a common point of view, such seemingly different things as, say, integral equations, systems of linear equations, and certain variational problems. This again means that functional analysis suggests common patterns for a certain circle of problems, and, as a result, common methods for their study.

An instructive example of this connection of functional analysis with other areas of mathematics is the small book of the Fields Medal laureate V. F. R. Jones, *Subfactors and knots* [**4**]. Try to separate there functional analysis (to be more specific, operator algebras), group theory, low-dimensional topology (knots and links), and quantum statistical mechanics!

(Unfortunately, lack of time and space prevented us from paying much attention to external connections and applications of functional analysis. Only now and then, if the temptation to speak about the "physical meaning", or "physical context" was too strong, we allowed ourselves to include some notes of informal or semi-literary nature. This concerns, for instance, the Riesz–Fischer theorem or the theorem on non-emptiness of a spectrum.)

On some principles for the selection of material. So you have a new textbook on functional analysis in your hands, a book intended for a first acquaintance with the subject. Of course, the author's duty is to say a few words about the specific features of the book, the selection of material, and the style of the exposition. Indeed, if the book does not differ from the other ones, why do we need it?

In this text we "grant civil rights" to some notions, methods, and results of modern functional analysis that are absent, or regarded as marginal in other textbooks. There is no need to list them: every specialist will see this from the contents. We shall only mention some main aspects.

Perhaps the main idea is that our book is written from the categorical point of view. Everywhere we stress and comment on the categorical nature of the fundamental constructions and results (like the constructions of adjoint operators and completion, the Riesz–Fischer and the Schmidt theorems, and closer to the end of the book, the great Hilbert spectral theorem). This, as we believe, provides a new level of understanding of the topics discussed. We are sure that students (and even professors!) are ready for the perception of the very basic categorical notions (and only those are used) and, what is more important, for the unifying mathematical language of category theory. Functional analysis, with its synthetic algebraic and topological content, works very well for first acquaintance with categories, the same way as "Analysis III" did for the exposition of the foundations of set theory 50 years ago. (Of course, the exposition must be accompanied by a sufficient supply of examples and exercises; but we shall discuss this later.)

As for specific elements of the modern techniques of analysis, we would like to distinguish tensor products of Banach spaces. We discuss two of them in detail, the "projective" and "Hilbert" tensor products, because one cannot work without them either in the geometry of Banach spaces, or in quantum statistical mechanics, or in the theory of elementary particles.

On some principles of the exposition.

1^0. *What we expect to be known.* We assume that the reader has mastered the material usually given in the first two years at mathematics departments of Russian universities (we used Moscow State University as an example). In particular, and this indeed is very important, the reader should know linear algebra, the foundations of real analysis (namely, Lebesgue measure and integral), and the elements of the theory of metric spaces, usually presented in advanced analysis courses. As for complex analysis, in my time, the corresponding lectures were given in the third year, then in the second, then in the third again. (The creative initiatives of the administration in this area have not settled down yet.) Fortunately, we will need complex analysis only in the middle of the book, when dealing with spectra (the Liouville theorem appears first). By that time the necessary lectures on complex analysis at Moscow State University will be delivered in any case.

The situation with topology and also with some topics in algebra (like tensor products of linear spaces) is more delicate. Formally, this material is a part of the second year syllabus. But, as our experience shows, it is dangerous to rely upon the corresponding obligatory courses where these questions are considered: usually the lecturers have in mind the goals that are too far from functional analysis. This is why in the book we give an independent exposition of the necessary topological and algebraic results.

2^0. *Standard and small print.* The text in ordinary print roughly corresponds, in our opinion, to the material of the course of functional analysis for third-year students of mathematical departments at Russian universities. We mean the full one-year course of functional analysis (four hours a week). The text in ordinary print is addressed to all students, no matter what their future mathematical specialty will be.

But for advanced students, who chose pure mathematics (algebra, geometry, or analysis) as their area of research, this large print is not sufficient, in our opinion. So we ask them to make themselves familiar with the small print ("noblesse oblige"). (Thus, you need not necessarily know how to decipher exact sequences of Banach spaces, but you must if you are to become a professional mathematician.)

Moreover, with the aim of satisfying possible curiosity, we also give in small print some material which is not necessary even for advanced students. Here we mean the information about quantum functional analysis, the structure of (co)products in some categories, etc. But in such cases we always let the reader know that the material is optional.

Needless to say, the text in ordinary print does not depend on the text in small print.

3^0. *About examples.* Possibly, we give more consideration to examples than it is usually done. When we introduce a new notion, we immediately collect a list of examples for the reader. For instance, we have a list of functors, a list of polynormed spaces, or a list of operators (the largest and the most important one). The reader should keep these lists ready at all times: whenever a new construction, property, or invariant appears, the reader should take examples from the lists and see which concrete form this construction acquires in these examples. (You can see how we do this with spectra of operators in Chapter 5.) We are sure this "playing with examples" is the only way for informal understanding.

4^0. *About exercises.* This book will not teach you much if you do not do the exercises.

Of course, a reader would usually like to postpone working on exercises more or less indefinitely. This is why exercises are included directly in the text. When you encounter an exercise, stop and do it (prove the assertion) before continue with reading. As a rule, our exercises are elementary (taking into account the hints). They illustrate the "main" text and make understanding less formal. Often they give a useful supplement to the proven proposition. There are, however, some more difficult exercises marked by the asterisk (*); here do what your conscience demands. On the contrary, the simplest exercises are marked by a zero (^{0}); they are absolutely necessary. Incidentally, our exercises are never used in the proofs of theorems, or in examples, but this does not mean, of course, that you can ignore them. On the other hand, exercises have their own hierarchy: the results of some of them can be used in others.

5^0. *Theorems presented without proofs.* The number of such theorems in this book is greater than usual. As a rule, these are results of great importance: "named theorems" that have simple and spectacular formulation, but whose proofs are relatively complicated and/or are based on facts exceeding the knowledge of our expected reader. So it is very desirable, moreover, in our opinion, necessary, to know these facts, but at the moment it is better not to waste time on analysing the proofs. Typical examples are the Enflo–Read theorem, the Milyutin theorem, and (above all) the Gelfand–Naimark theorem. For all these theorems, a reference where the proof can be found is provided.

6^0. *Technical details.* The book is divided into chapters and sections; a reference such as "see Section 0.6" means Section 6 in Chapter 0.

We distinguish and number independently the following types of mathematical statements: definitions, theorems, propositions, corollaries, examples, and exercises. (There are also "remarks" and "warnings", which are not numbered.) When we refer to, say, Proposition 1.2.3 (correspondingly, Proposition 2.3, Proposition 3), we have in mind Proposition 3 of Section 2 in Chapter 1 (correspondingly, Proposition 3 of Section 2 in the current chapter, or Proposition 3 in the current section.)

The end of a proof is marked by the sign ■. Some of theorems and propositions in the book are given without proof. The "end of proof" sign placed immediately after a statement means that the proof of the statement is clear or can be easily verified. If the sign is absent, then the proof of the result can be found in the supplied reference(s). Usually, these are important theorems for which, we believe, the reader should know the statements (see the previous remark). Corollaries are assertions that immediately follow from what was proved earlier.

The symbol $\Longleftrightarrow$ means "if and only if". The combination := means "by definition".

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So these are our good intentions. But, of course, the readers themselves will judge whether we carried them out successfully.

Acknowledgements. While writing this book I frequently asked my colleagues—experts in different fields of functional analysis—various questions. In particular, A. M. Stepin and O. G. Smolyanov expressed their enlightened opinions on many questions.

I appreciate very much the help of A. Yu. Pirkovskii, my friend and former student, who practically played the role of the editor of this book. If we managed to make the set of errors and misprints become nowhere dense instead of everywhere dense, as was initially the case, this is entirely to his credit. In addition, Pirkovskii suggested some useful amendments to the text, and in particular, enriched it with a series of useful exercises.

I would like to thank my friend S. S. Akbarov, who made a series of critical remarks about the book.

Finally, I would like to acknowledge my gratitude to the Moscow Center for Continuous Mathematical Education for their suggestion to write this book, and to the Russian Foundation for Basic Research for their financial support.