Preface

In this book we describe the elementary theory of operator algebras and basic tools in noncommutative geometry.

In his early work on noncommutative geometry, A. Connes proposed to treat a general noncommutative C^*-algebra as the C^*-algebra of "continuous functions on a noncommutative space". The study of interactions between topology/geometry and analysis via algebraic objects originates with I.M. Gel'fand. According to Gel'fand's theory, the topological structure of a compact topological space X is completely determined by the algebra $C(X)$ of continuous functions. Even a smooth structure of a differentiable manifold can be captured by an algebraic object. Pursell's theorem says that two compact smooth manifolds are diffeomorphic if and only if the $\mathbb{R}$-algebras of smooth functions are isomorphic.

It may be natural to consider $C^k(M)$ $(k = 0, 1, 2, \cdots, \infty)$ for a given C^∞-manifold. These are infinite-dimensional vector spaces. In order to control "infinite dimension", we need to study these spaces with topology. For a finite k the space $C^k(M)$ has the structure of a Banach space, and $C^\infty(M)$ is a Fréchet space. Since we study linear algebras (*e.g.* eigenvalue problems) of infinite dimension, it would be suitable to consider algebras over $\mathbb{C}$. Hence from now on $C^k(M)$ denotes $\mathbb{C}$-valued C^k-class functions. Complex conjugation of $\mathbb{C}$-valued functions defines a C^*-algebra structure on $C(M) = C^0(M)$, a Banach $*$-algebra structure (§1) on $C^k(M)$ $(0 < k < \infty)$ and a Fréchet $*$-algebra structure on $C^\infty(M)$.

Generally speaking, C^*-algebras are more well behaved than Banach $*$-algebras, and beautiful theories have been established. Once we accept the Gel'fand correspondence of topological spaces and abelian C^*-algebras as natural, nothing keeps us from investigating noncommutative C^*-algebras which correspond to "singular" spaces such as the leaf spaces of foliations. Accordingly, it is important to learn not only about commutative C^*-algebras but also about noncommutative C^*-algebras.

In a sense noncommutative geometry is a geometry of "virtual spaces" or "pointless spaces". However, that may be misleading. Noncommutative geometry should be thought of rather as a paradigm than as a theory. The core idea is to express geometry as an operator on the representation space of an algebra. As it turns out, noncommutative geometry provides unification of various mathematical concepts, *e.g.* spin geometry, geometry of fractals, geometry of discrete groups, pseudo-differential calculus, and so on.

In the early 1980's topologists and geometers for the first time came across unfamiliar words like C^*-algebras and von Neumann algebras through the discovery of new knot polynomials by V.F.R. Jones or through S. Hurder's remarkable result on the relationship between the vanishing of the Godbillon-Vey classes for foliations and the types of foliation von Neumann algebras. During the following decade, a great deal of progress in the area of interaction between geometry and analysis was achieved. We list just a few developments: cyclic cohomology theory, KK-theory, applications of operator algebras to the Novikov conjecture on homotopy invariance of higher signature in topology.

Geometers in Japan organized a workshop in 1998 to learn operator algebras and its applications to other fields. At that time when topologists/geometers wanted to study the theory of operator algebras, not much suitable material was available. Of course, there were many good books, but they were mostly aimed at those who had a thorough knowledge of functional analysis. This book was initially prepared for the workshop "Surveys in Geometry" held in the fall of 1998 at the University of Tokyo, and it is aimed at topologists and geometers, with less background in analysis. We shall provide an overview of operator algebra theory and explain basic tools used in noncommutative geometry and finally applications to Atiyah-Singer type index theorems. Our purpose here is to convey an outline and general idea of the theory of operator algebras, to some extent focusing on examples. To that end some details and proofs will be omitted. Hence, we give a list of references that can be easily obtained. For those who do not care about details of proofs, [**46**] is easy reading. For those who care about some detail, [**80**] may be suitable. Finally, for those who want to thoroughly understand the proofs, [**67**] and [**109**] are excellent. Additional reading material will be referred to in each chapter.

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