Preface

This book presents a study of quantum analogs of bounded symmetric domains. The latter domains are steady subjects of attraction for specialists in geometry, algebra, and analysis, basically as sources of exactly solvable problems of complex analysis, noncommutative harmonic analysis, and classical mathematical physics.

The simplest bounded symmetric domain is the unit disc $\mathbb{D} = \{z \in \mathbb{C} | \, |z| < 1\}$. Its quantum analog was introduced by S. Klimek and A. Lesniewski [**173**]. We find it reasonable to start with Chapter 1 on the quantum disc. Our motive here is to avoid distracting the reader by algebraic details, but instead to produce an outline of the problems of noncommutative complex and harmonic analysis in which we are interested. This chapter can be used as background material to a seminar for university students in mathematics.

The problems discussed in Chapter 1 admit a reformulation (many of them, even a solution) in a much more general context, namely within the framework of quantum theory of bounded symmetric domains [**289, 290**], whose basics are expounded in Chapter 2. This demonstrates an interplay between the theory of quantum groups and noncommutative complex analysis.

A more detailed view of Chapters 1 and 2 is given in the table of contents.

The author's intention for subsequent chapters was to present results of O. Bershtein, Ye. Kolisnyk, D. Proskurin, D. Shklyarov, S. Sinel'shchikov, A. Stolin, L. Turowska, L. Vaksman, and G. Zhang, [**33, 36, 34, 308, 258, 284, 320, 279, 318, 281, 287**], together with unpublished results on quantum bounded symmetric domains (some of those are already present in `xxx.lanl.gov`, see, e.g., [**317**], or will appear therein in the nearest future).

Regretably, this plan is not accessible due to some nonmathematical reasons. Instead, we are going to form a Web page dedicated to quantum bounded symmetric domains, which will contain a draft of the conjectured full version of the book. It is expected to be twice as large as the present volume; it will, in particular, contain a discussion on unsolved problems.

I am deeply grateful to my students O. Bershtein, Ye. Kolisnyk, L. Korogodski, D. Shklyarov, and coauthors Ya. Soibelman and A. Stolin. Also, my special thanks to H. Jakobsen, A. Klimyk, E. Koelink, S. Kolb, Yu. Samoilenko, K. Schmüdgen, L. Turowska, and G. Zhang for numerous helpful discussions on the results of this book.

A special role in my life was played by Vladimir Drinfeld. In the mid-1980s he taught me the basics of the theory of quantum groups and helped me to return to mathematics after an involuntary break I had to take from it for many years.

Here is some historical background. In the late 1970s, a study of exactly solvable problems of statistical mechanics and quantum field theory led L. Faddeev and his team to the creation of the quantum method of inverse scattering problems [**293,**

294, 305]. They introduced the quantum Yang-Baxter equation and associated its solutions to series of exactly solvable problems of mathematical physics. Their work [**206**] presents the solutions of this equation which were known before 1980; the authors mention that "its deep relations to the mathematical fields like group theory and algebraic geometry are coming into the picture". In the early 1980s plenty of literature was dedicated to bringing to light those relations and studying solutions of the quantum Yang-Baxter equation. In this context, we mention the works of E. Sklyanin, P. Kulish, V. Drinfeld, and N. Reshetikhin [**291, 204, 77, 292, 205**].

In 1984 V. Drinfeld introduced quantum analogs for universal enveloping algebras (see Subsection 2.1.1), and his talk on quantum groups at the Gelfand seminar became a crucial point in the theory of quantum groups. Independently, in 1985 M. Jimbo came to his version of quantum analogs of universal enveloping algebras. The crucial works in quantum group theory are [**78, 140**] as well as survey reviews [**79, 141, 94**].

A different approach to quantum groups is due to S. Woronowicz [**341, 342**]. This approach was used later in the theory of compact quantum groups [**344, 72, 66**], where the simplest example is the quantum group $SU(2)$ [**343, 251, 321, 228, 239, 240, 229, 191, 193, 170, 316, 37, 152, 153, 184, 189, 182, 180, 252, 254, 213, 1**].

Applications of the theory of quantum groups in low-dimensional topology and category theory as well as in conformal quantum field theory have been found later; see [**262, 307, 160, 150, 223, 222, 5, 247**]. In this book we do not consider these and many other applications, and we restrict ourselves to quantum analogs of homogeneous spaces of noncompact real Lie groups.

The initial results in this direction were obtained in late 1980s in [**319**]. In this work the quantum group of motions of the Euclidean plane was introduced; then it became a tool for solving a number of problems in noncommutative harmonic analysis and special functions [**56, 346, 347, 59, 217, 194, 183, 181, 187, 188, 186, 301, 38, 2**]. The quantum group of motions of plane can be derived from the quantum group $SU(2)$ by an application of the Inonu-Wigner contraction [**326**, p. 234]; this trick is also applicable to some other "nonhomogeneous" quantum groups [**58, 57, 75**].

P. Podleš and S. Woronowicz managed to produce a quantum analog for the Lorentz group [**255**] by applying another method to real forms of complex semisimple Lie groups. Their research has been succeeded in [**259, 253, 244, 243, 52, 51**].

Essential obstacles have been encountered on the way to a quantum analog of the group of motions of the Lobachevski plane (more precisely, to a quantum analog of some locally isomorphic group). It even happened that S. Woronowicz denied the very existence of this quantum group [**345**]. Nevertheless, he changed his opinion later, possibly after reading the work of L. Korogodsky [**200**]. Several years later, a construction of the required quantum group was completed by E. Koelink and J. Kustermans [**179**] within the Kustermans-Vaes axiomatics [**207, 208**].

The above references are not exhaustive; however, they provide a general idea about the development of noncompact quantum group theory at its earliest stage.

An apparent feature of [**200**] is a sharp discrepancy between the simplicity of the classical subject and the complexity of its quantum analog. This inspired an idea of not using function algebras on quantum groups when studying function algebras on quantum homogeneous spaces. Under this approach, the methods of representation theory of real reductive Lie groups could be the principal tools of research [**332, 333**]. In [**114**] Harish-Chandra introduced the holomorphic discrete series of representation for groups of Hermitian type. These representations are realized in weighted Bergman spaces of holomorphic functions on bounded symmetric domains [**197**]. The spaces of the associated action of Lie algebras are polynomial algebras on prehomogeneous complex vector spaces of commutative parabolic type (see Subsection 2.3.8). In this context it was natural to ask if there exist quantum analogs of bounded symmetric domains, weighted Bergman spaces, holomorphic discrete series, and the Sato–Bernstein polynomials of prehomogeneous vector spaces of commutative parabolic type (see [**234, 268**] for a background on Sato–Bernstein polynomials).

The positive answer to this question was supposed to lead to substantial progress in the theory of quantum prehomogeneous vector spaces and Harish-Chandra modules [**270**] over quantum universal enveloping algebras. This could result in a breakthrough in noncommutative complex analysis, a field coming back to a work by W. Arveson [**17**]. It became clear later that it was the right idea.

In late 1990s three teams of mathematicians obtained initial results in quantum theory of bounded symmetric domains. These teams acted independently, being unaware of each other and thinking of their methods as self-sufficient.

T. Tanisaki and his collaborators introduced q-analogs of prehomogeneous vector spaces of commutative parabolic type and found an explicit form of the associated Sato–Bernstein polynomials [**156, 155, 233, 154**].

H. Jakobsen then suggested an easier way of producing these quantum vector spaces; he also realized that he was working toward quantum Hermitian symmetric spaces of noncompact type [**136, 138**].

The authors of the papers listed above overlooked the so-called hidden symmetry of the quantum prehomogeneous vector spaces in question [**288, 282**]; perhaps, it was the reason why they made no crucial step on the way to quantum bounded symmetric domains.

Foundations of the quantum theory of bounded symmetric domains were laid in [**289**]. The compatibility of the approaches used in [**306, 136, 289**] has been established by D. Shklyarov [**276**].