Preface

The source of the charm of number theory is the wonder of prime numbers. To elucidate the mystery of prime numbers, number theorists have made various approaches. In *Number Theory 1 and 2*, we saw ζ functions and class field theory as examples of such approaches. In this volume we continue to introduce fundamental methods of modern number theory.

One characteristic of modern number theory is the intertwinement of algebra and analysis. Algebraic entities are Galois groups and algebraic geometric objects, and analytic entities are ζ functions, modular forms, and automorphic representations. For example, the heart of class field theory, established by Teiji Takagi, may be expressed as follows: The one-dimensional representation of Galois group, an algebraic entity, and the one-dimensional representation of idele class group (Hecke character), an analytic entity, have the same ζ function. In this volume we introduce Iwasawa theory, in which an analytic entity called a p-adic L-function, a p-adic incarnation of ζ function, appears. Its algebraic and arithmetic significance will be revealed.

A generalization of class field theory to the case of nonabelian Galois groups, called "nonabelian class field theory", is one of the main themes of modern number theory which is still under construction. Its first example is the correspondence between elliptic curves, an algebraic entity, and modular forms of congruent subgroups of the modular group, an analytic entity. By establishing this correspondence Andrew Wiles proved Fermat's Last Theorem in 1995, more than 350 years after it was first proposed.

With these trends in modern number theory as a background, we introduce the fundamentals of the theory of modular forms and Iwasawa theory. We also describe the arithmetic of elliptic curves by giving an outline of the proof of Fermat's Last Theorem by Wiles.

Each chapter provides explicit calculations of examples to enhance understanding. We hope readers will compute several examples and equations by themselves so that they can experience modern number theory.

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Preface to the English Edition

This is the English translation of the Japanese book *Number Theory 3*, the third of three volumes in the "Number Theory" series. The original Japanese book was published in 1998 (the second edition in 2005). Instead of Kazuya Kato, who co-wrote "*Number Theory 1 and 2*", Masato Kurihara is the co-author of *Number Theory 3*.

In this volume, we study modular forms and Iwasawa theory, which are very important subjects in modern number theory. (See the Objectives and Outlines of These books section of this book for the details.) As in *Number Theory 1 and 2*, we explain these theories with many concrete examples for non-specialists and beginners. In the final chapter we begin with the basics of the arithmetic of elliptic curves and give a brief exposition of a proof of Fermat's Last Theorem by Wiles. The authors hope that readers enjoy the wonderful world of modern number theory.

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