Preface

Arakelov geometry is one of the branches in arithmetic geometry. Nevertheless, its research area is very widespread. Algebra, geometry and analysis are all part of it. It is also closely related to theoretical physics. To give full expositions of Arakelov geometry in one book is almost impossible. Fortunately, the arithmetic Riemann-Roch theorem, which is one of the most significant theorems in Arakelov geometry, is treated in the books of Soulé [**64**] and Faltings [**20**], so this book is mainly devoted to a field of the so-called "birational Arakelov geometry". First, let us recall a brief history of Arakelov geometry.

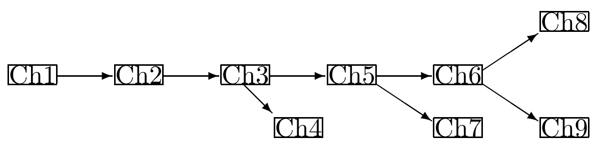
Diophantus at Alexandria might be the founder of Arakelov geometry, as its source is Diophantine geometry. However, its concrete form has appeared in Arakelov's paper [**2**] in 1974. The intersection theory and the Riemann-Roch theorem may be the most important tools in the theory of algebraic surfaces. Due to them, Arakelov actually made significant progress on the problems of rational points over function fields. In this sense, it was very natural to seek an analogue of the intersection theory and the Riemann-Roch formula on arithmetic surfaces. A prototype of his idea is the geometry of numbers due to Minkowski (see Chapter 2). An important point of Arakelov's idea is to treat a height function in terms of intersection numbers (see Section 9.1). In this sense, the Mordell-Weil theorem from the viewpoint of the height function is also a prototype of his plan. In the paper [**2**], Arakelov had partially succeeded his dream, but it was not complete. The full treatment was done in Faltings' paper [**18**]. In his paper, Faltings established the fundamental results on arithmetic surfaces, such as the arithmetic Riemann-Roch formula, the arithmetic Noether formula and so on.

The next problem is a generalization to a higher dimensional arithmetic variety. The first one is the intersection theory in a higher dimensional case. Gillet-Soulé [**23**] solved it by considering Green currents instead of Green functions. The second one is the arithmetic Riemann-Roch formula including the arithmetic Noether formula. As treated in this book (see Section 4), the arithmetic Riemann-Roch formula on arithmetic surfaces due to Faltings, however elementary, is very complicated, so that it seemed to be difficult to generalize it to the higher dimensional case. Despite this, Deligne [**14**] reported a treatment of the arithmetic Riemann-Roch formula in terms of the Quillen metrics of determinant bundles (it was also pointed out in Faltings' paper [**18**]) and he built it up on arithmetic surfaces. The key for his proof is the usage of the anomaly formula in string theory. Soon after the generalization of intersection theory, Gillet-Soulé [**26**] succeeded again in proving the arithmetic Riemann-Roch theorem on a higher dimensional arithmetic variety. In this way, the fundamental tools for Arakelov geometry had been established.

A typical application of the arithmetic Riemann-Roch theorem is the existence of small sections, that is, sections whose norm is less than or equal to 1. It was used in the solution of the Mordell conjecture due to Vojta [**71**]. Moreover, Ullmo [**70**] and Zhang [**77**] gave an affirmative answer to the Bogomolov conjecture using Arakelov geometry. The existence of small sections was also crucial for their solution. Why are the small sections important? In algebraic geometry, it is a very important problem to find global sections of an invertible sheaf, or to determine the dimension of the vector space consisting of global sections. In this sense, it is significant to require an arithmetic analogue of global sections. As explained in Remark 5.2, a section with small norm on an arithmetic variety means an arithmetic analogue of a global section and the logarithm of the number of small sections is nothing more than the dimension in the arithmetic sense. In the geometric case, it is sufficient to consider non-archimedean norms, but archimedean norms turn out to be important on an arithmetic variety. Of course, an archimedean norm is not determined uniquely. In Arakelov geometry, choosing a different norm is viewed as a different compactification, so that we always fix a metric of an invertible sheaf. For this reason, complex geometry plays an important role in Arakelov geometry. In this book many pages are devoted to the arguments of complex geometry.

In algebraic geometry, to study the asymptotic behaviour of powers of an invertible sheaf is nothing more than the study of birational geometry. Similarly the purpose of birational Arakelov geometry is to consider an arithmetic analogue of them. In this book, we try to include the recent results of birational Arakelov geometry, like the continuity of the arithmetic volumes, the generalized Hodge index theorem, and so on.

The following is the flowchart of this book.



Chapter 1 is devoted to preliminaries for all chapters. In Chapter 2, the fundamental results of geometry of numbers are discussed. In Chapter 3, we give expositions of Arakelov geometry on arithmetic curves, which will be an introduction to Arakelov geometry. In Chapter 4, we consider Arakelov geometry on arithmetic surfaces. This chapter is intended for beginners of Arakelov geometry and its context is not referred to in later chapters, so the reader can skip this chapter. Chapter 5 is devoted to the expositions of the basic results of Arakelov geometry on a general arithmetic variety. In Chapter 6, the recent result of birational Arakelov geometry on the continuity of the arithmetic volumes is discussed. In Chapter 7, we treat the arithmetic Nakai-Moishezon criterion, which is also the fundamental theorem of birational Arakelov geometry. Chapter 8 is devoted to the observation of the arithmetic Bogomolov inequality. In Chapter 9, we consider the Lang-Bogomolov conjecture as an application of birational Arakelov geometry. In this translation, the following points are different from the the original Japanese book:

(1) I adopt a systematic approach to establish the arithmetic Nakai-Moishezon criterion.
(2) I omit the postscripts at the end of each chapter because their contexts are out of date.

I expect the reader to be familiar with the basic algebraic geometry including the scheme theory. The prerequisite for reading this book is, for example, the context of Hartshorne's book [**29**] and the first half of the Griffiths-Harris book [**27**] on complex geometry. I hope the reader will see the attraction of Arakelov geometry throughout this book.

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