Preface

This book is an extended version of the notes of a reading seminar “Arithmetic methods in algebraic geometry” run by the authors at the Steklov Mathematical Institute in Spring 2011. The goal of the book is to give an introduction to the theory of unramified Brauer groups and their applications to stable rationality, starting with the most basic concepts of group cohomology. For this reason, we omit many popular topics that are already well covered in textbooks (Galois cohomology, Brauer groups, etc). Instead we give more attention to applications of unramified Brauer groups to stable non-rationality, and to an example of a non-rational stably rational variety. As far as we know, these topics are not covered in detail in standard textbooks, and the proofs presented in the original sources require substantial effort to understand.

The style of our seminar suggested delivering the material through problems and exercises. We have tried to split the proofs of all facts that we need into relatively simple steps and provide detailed hints for all non-trivial points. This gives us hope that studying our book will be no more difficult (or at least not much more difficult) than reading a usual textbook, not to mention research articles. Most of the book is accessible to those who are familiar with basic algebra, Galois theory, and fundamental notions of algebraic geometry.

In Chapter 1 we collect the necessary definitions and facts concerning cohomology of abstract groups. The same is done in Chapter 2 for Galois cohomology and in Chapter 3 for Brauer groups. Since the significance of these topics is much broader than their applications to stable rationality and they constitute an important part of modern mathematical culture, we recommend that the reader continue their study with the help of canonical sources. For group cohomology we recommend Chapter IV in the book [**CF67**], for Galois cohomology the book [**Ser65**] and Chapter V in [**CF67**], and for Brauer groups Chapter X of the book [**Ser79**] and [**Bou58**]. In Chapter 4 we focus on the Brauer group of a discrete valuation field and in particular define the unramified Brauer group. For further reading on these topics, we refer to the book [**Ser79**] and §1 of Chapter VI in the book [**CF67**]. Besides these references, most of the material of Chapters 1–4 is covered in much more detail in the textbook [**GS06**]. The interested reader can also find an accessible account of Galois cohomology in [**Ber10**].

In Chapter 5 we present the example of a quotient variety $X = V/G$, where G is a finite group and V is a representation of G over an algebraically closed field k of characteristic zero, which can be proved to be non-rational (and even not stably rational) using the notions introduced earlier. The obstruction we use is non-triviality of the unramified Brauer group of the field $k(X)$, that is, of the invariant field $k(V)^G$. Examples of this kind first appeared in the works of

D. Saltman [**Sal84**] and F. A. Bogomolov [**Bog87**], but we adopt the simpler approach taken by I. R. Shafarevich in [**Sha90**]. The variety X has relatively large dimension; one may be interested in whether there are similar examples in lower dimensions. It turns out that this is possible already for some threefolds, based on a completely different construction from the one given in Chapter 5. We present such an example in Chapter 7: the well-known construction of a non-rational singular double cover of $\mathbb{P}^3$ branched over a quartic. This variety was first described in the paper [**AM72**] by M. Artin and D. Mumford, but we take a more algebraic approach due to M. Gross (see [**AM96**, Appendix]). Before presenting the construction, we introduce the Clifford invariant and spend some time on auxiliary results about quadrics over non-algebraically closed fields in Chapter 6. More on quadrics over non-algebraically closed fields can be found in the book [**EKM08**]. In Chapter 7 we also provide a unirationality construction for a double cover of $\mathbb{P}^3$ branched over a quartic (here we mostly follow the proof of Theorem IV.7.7 in the book [**Man86**]). A detailed survey of stable (non-)rationality results for quotient varieties similar to the ones considered in Chapter 5 is contained in [**CTS07**]; one can also find references to many original works on the topic therein. We also recommend that the reader have a look at the short survey [**BT17**]. For a discussion of results on obstructions to stable rationality appearing from the Artin–Mumford construction, we refer the reader to the survey [**Pir16**].

In Chapter 8 we introduce Weil restriction and discuss its main properties, and we also establish some properties of algebraic tori that will be used in Chapter 9. More details on algebraic tori are available in the book [**Vos98**]. Chapter 8 also discusses the notion of universal torsor and some basic properties of Châtelet surfaces. Since we already know examples of varieties which are not stably rational, it is natural to ask whether or not stable rationality is actually the same as rationality. It turns out that it is not the same, but producing an example that separates these two concepts is not easy at all. This is done in Chapter 9, following the paper [**BCTSSD85**] by A. Beauville, J.-L. Colliot-Thélène, J.-J. Sansuc, and P. Swinnerton-Dyer. At the end of Chapter 9 we provide an argument of N. Shepherd-Barron from [**SB04**] that slightly enhances the construction of [**BCTSSD85**]. Throughout Chapter 9 we try to use geometric language and to avoid coordinates and explicit equations as far as possible, which we hope will make our exposition a bit more transparent than that of [**BCTSSD85**] and [**SB04**].

Chapters 10 and 11 are devoted to one more application of unramified Brauer groups, namely, to Brauer–Manin obstructions. The main purpose of Chapter 10 is to provide some motivation for this: we discuss a proof of the classic Minkowski–Hasse theorem for quadrics (to be more precise, we deduce this theorem from the fundamental facts of class field theory). Our exposition mostly follows Chapter IV of the book [**Ser70**], but we try to use more geometric language when reducing the multi-dimensional case to the one-dimensional case. In Chapter 11 we define the Brauer–Manin obstruction and use it to produce a counterexample to an analog of the Minkowski–Hasse theorem for curves of genus 1. More on the Brauer–Manin obstruction can be found in the surveys [**Sko01**], [**Poo17**], and [**Wit16**] (see also the brief exposition in [**MP05**, 5.2.3]).

Appendix A contains a collection of references to the main results on étale cohomology that are necessary for interpretation of Brauer groups in étale terms (see [**Dan96**] or [**Mil80**] for more details on these results). Those who have a taste for exploring primary sources may wish to take a look at the text [**Gro95b**] by A. Grothendieck, where this very approach was used to introduce the unramified Brauer group for the first time.

As one might expect, we were not able to pay enough attention to many topics related to unramified Brauer groups (in particular, to the study of stable rationality, which became remarkably active in recent years). To (partially) fill this gap, we conclude most of the chapters of the book with lists of references for an interested reader, sometimes with brief explanations about their connections to the material covered in the chapter. In several cases (especially in Chapters 5, 7, and 11) we also tried to include references to recent works, because the techniques mentioned in these chapters are still being developed and actively applied.

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Sergey Gorchinskiy and Constantin Shramov