Introduction

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1. The subject

Interactions between algebraic and analytic techniques in the study of complex algebraic varieties go back to the beginnings of the twentieth century. However, over the past twenty years these connections have particularly flourished. Analytic methods and ideas around multiplier ideals have been successfully introduced in the study of higher-dimensional algebraic varieties, and tools developed as part of the Minimal Model Program have had growing impact. Taken together these new methods achieved great success, culminating in the proof of a fundamental problem in the field, the finite generation of the canonical ring of an algebraic variety. While some of the analytic techniques have already found an algebraic counterpart and vice versa, much is left to be done, and it is expected that we will see more of this interaction in the future.

The 2008 PCMI Summer School was centered around these exciting new developments at the crossroads of analytic and algebraic geometry, and in particular, on the two existing approaches to the finite generation of the canonical ring. The program had several components. There was a Graduate Program, consisting of eight mini-courses, half of which were devoted to analytic, and respectively, algebraic topics. The lecturers were Bo Berndtsson, John D'Angelo, Jean-Pierre Demailly, Christopher Hacon, János Kollár, Robert Lazarsfeld, Mircea Mustaţă, and Dror Varolin. In parallel with this there was an active Research Program, organized around one or two daily seminar talks. The research environment was enhanced by the presence of the Clay Senior Scholar Robert Lazarsfeld, and of the Program Principal Yum-Tong Siu. They each gave a public lecture, introducing some of the main questions in analytic and algebraic geometry.

This volume consists of the contributions of the eight lecturers in the Graduate Program. In addition, it contains two expository presentations introducing the finite generation of the canonical ring, from the two different perspectives.

2. Content of the volume

The mini-courses consisted of five lectures each, and three additional problem sessions run by TA's. Each sequence of four mini-courses, on the algebraic and on the analytic side, was organized as to give a gradual introduction to the recent developments. We have decided to keep the same order for the contributions in this

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volume. At the end of each of the two sequences, we have included one expository paper that was not based on the lectures in the PCMI program. We hope that in terms of both topic and presentation, these additions fit well with the rest of the volume.

As expected, there was a lot of interaction between the two sequences of courses, and this is transparent in the lecture notes in this volume. Some key concepts (most notably, multiplier ideals) and results (such as vanishing and lifting theorems) have played an important role in the cross-fertilization of analytic and algebraic geometry, and especially in the recent applications to birational geometry. We hope that seeing these notions and results developed in both contexts, will give the reader a sense of the close interaction between these two fields.

The contributions on the analytic side are the following.

An Introduction to Things $\bar{\partial}$. These are the lecture notes from Bo Berndtsson's course at PCMI 2008. The aim of this course was to give an introduction, appropriate for a beginning student in complex analysis, to the basic analytic techniques behind weighted L^2 estimates for the $\bar{\partial}$-equation. The notes here reflect this aim by starting with the $\bar{\partial}$-equation in one-variable, carefully examining the fundamental issues in this special case, and then seeing how these issues are handled in ever greater generality, i.e. for domains in several variables, on Stein manifolds, and for bundle-valued forms over general complex manifolds. The emphasis throughout the notes is on how the basic L^2 estimates for $\bar{\partial}$ change as additional structure (primarily metrics on both the base manifold and on the bundles) is added. Several applications of L^2 estimates on $\bar{\partial}$ are also presented, including vanishing theorems of Kodaira! type and various extension results connected to the Ohsawa-Takegoshi theorem.

Real and Complex Geometry Meet the Cauchy-Riemann Equations. In his course at the Summer School, John D'Angelo discussed several analytic topics that involve an interplay between real and complex geometry. The order of contact between a complex analytic variety and a real hypersurface inside a complex manifold, and its relation to estimates for the $\bar{\partial}$ operator, forms the basis for the first half of these notes. An expanded discussion of Kohn's subelliptic multipliers for the $\bar{\partial}$-Neumann problem, which inspired the general notion of multiplier ideals studied in algebraic and analytic geometry, is also given. A detailed presentation of a version of Hilbert's 17th problem, and a sketch of its relationship with special metrics over the complex projective space concludes these notes.

Three Variations on a Theme in Complex Analytic Geometry. This chapter is based on Dror Varolin's lecture series. These notes show how L^2 methods are used to obtain three foundational results in analytic geometry: Kodaira's embedding theorem, the L^2 holomorphic extension theorem of Ohsawa-Takegoshi, and Skoda's division theorem. The connection between curvature, in several forms, and the estimates on $\bar{\partial}$ used to obtain these results is emphasized throughout the notes. Multiplier ideal sheaves are also discussed, from the analytic perspective, and proofs of Nadel's vanishing theorem and Siu's theorem on the global generation of these ideals are presented.

Structure Theorems for Projective and Kähler Varieties. These are lecture notes based on Jean-Pierre Demailly's course. They examine vanishing results for $\bar{\partial}$ cohomology on compact Kähler manifolds (or more particularly, projective manifolds) that carry line bundles of various types, e.g. numerically effective, pseudo-effective, ample, or big line bundles. The positivity conditions expressed by these types of bundles is the central theme of these notes. Recent results on the deformation theory of curves in Kähler manfolds and approximation of closed, positive $(1,1)$-currents are discussed in some detail. The notes also give an outline of Siu's analytic approach to finite generation of the canonical ring and a sketch of Păun's related non-vanishing theorem.

Lecture Notes on Rational Polytopes and Finite Generation. Mihai Păun, who was unable to attend the PCMI 2008 program, graciously contributed this chapter in his absence. The notes present an overview of an analytic approach to the Birkar-Cascini-Hacon-M^{c}Kernan theorem on the finite generation of the canonical ring, using techniques pioneered by Siu, Demailly, and Shokurov. The polytopes in the pseudoeffective cone of a nonsingular projective variety were introduced by Shokurov, and they play an important role in this proof of finite generation (as well as in the original one) The technical core of the arguments relies on analytic methods, in particular on extension theorems established previously by Păun, which are discussed also in these notes. The flexibility of the analytic approach to finite generation of the canonical ring, as proposed by Siu, motivates much of the presentation.

The following are the contributions on the algebraic side.

Introduction to Resolution of Singularities. This is based on Mircea Mustaţă's lecture series. Assuming a minimum of background in algebraic geometry, it presents a proof of Hironaka's Theorem on resolution of singularities. Despite being arguably one of the most important results in algebraic geometry, for many years its proof has been understood by only a handful of experts. The combined work of many people has resulted in making the proof accessible to any algebraic geometry student. These notes give a full account of the proof, based on the recent simplifications of Włodarczyk and Kollár.

A Short Course on Multiplier Ideals. These are lecture notes of Robert Lazarsfeld's mini-course. Multiplier ideals have been central to many of the recent developments in higher-dimensional algebraic geometry. These notes give a broad overview of the algebraic theory of multiplier ideals. They cover the definition and basic properties of multiplier ideals, the connection with vanishing theorems that lies at the heart of the subject, as well as a wide range of applications. In particular, one describes the use of multiplier ideals in the context of lifting theorems, as pioneered by Siu in his proof of Invariance of Plurigenera.

Exercises in the Birational Geometry of Algebraic Varieties. In his PCMI course, János Kollár gave an introduction to the main ideas, results, and open problems in the Minimal Model Program. The contribution in this volume offers an introduction to the same topic through one hundred exercises. It contains the basic definitions, as well as the statements of and comments on the main results. The rest of the notes is devoted to illustrating through exercises and examples these results, the existing techniques in birational geometry, with their strengths as well as their shortcomings. They cover basic topics about birational maps, classical results about surfaces, the cone of curves, flips and minimal models.

Higher Dimensional Minimal Model Program for Varieties of Log General Type. These are lecture notes for the course given by Christopher Hacon during the Summer School. They present some of the recent results of Hacon-M^{c}Kernan and of Birkar-Cascini-Hacon-M^{c}Kernan leading to the proof of the finite generation of the canonical ring. They cover more advanced topics on multiplier ideals, and in particular, the lifting theorems that are then used in the proof of existence of flips. The notes end with a discussion of the Minimal Model Program with scaling, and with a sketch of the argument for the existence of minimal models for varieties of log general type.

Lectures on Flips and Minimal Models. In April 2007, MSRI organized the workshop "Minimal and Canonical Models in Algebraic Geometry", motivated by the recent progress in higher-dimensional birational geometry. These lecture notes of Alessio Corti, Paul Hacking, János Kollár, Robert Lazarsfeld, and Mircea Mustaţă are based on talks at this workshop, giving an informal overview of the work of Hacon-M^{c}Kernan and Birkar-Cascini-Hacon-M^{c}Kernan proving the finite generation of the canonical ring. In particular, these notes could be used as an introduction to the more detailed expositions of Lazarsfeld and Hacon in this volume.

3. Acknowledgments

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