Contents

Preface xiii

Jeffery McNeal and Mircea Mustaţă
Introduction 1

Bo Berndtsson
An Introduction to Things $\overline{\partial}$ 7

Introduction 9

Lecture 1. The one-dimensional case 11
1.1. The $\bar{\partial}$-equation in one variable 11
1.2. An alternative proof of the basic identity 14
1.3. An application: Inequalities of Brunn-Minkowski type 14
1.4. Regularity — a disclaimer 16

Lecture 2. Functional analytic interlude 19
2.1. Dual formulation of the $\bar{\partial}$-problem 19

Lecture 3. The $\bar{\partial}$-equation on a complex manifold 25
3.1. Metrics 25
3.2. Norms of forms 27
3.3. Line bundles 29
3.4. Calculation of the adjoint and the basic identity 33
3.5. The main existence theorem and L^2-estimate for compact manifolds 35
3.6. Complete Kähler manifolds 37

Lecture 4. The Bergman kernel 43
4.1. Generalities 43
4.2. Bergman kernels associated to complex line bundles 46

Lecture 5. Singular metrics and the Kawamata-Viehweg vanishing theorem 51
5.1. The Demailly-Nadel vanishing theorem 51
5.2. The Kodaira embedding theorem 54
5.3. The Kawamata-Viehweg vanishing theorem 55

Lecture 6. Adjunction and extension from divisors 59
6.1. Adjunction and the currents defined by divisors 59
6.2. The Ohsawa-Takegoshi extension theorem 62

Lecture 7. Deformational invariance of plurigenera 71
7.1. Extension of pluricanonical forms 71

Bibliography 75

John P. D'Angelo
Real and Complex Geometry meet the Cauchy-Riemann Equations 77

Preface 79

Lecture 1. Background material 81
1. Complex linear algebra 81
2. Differential forms 82
3. Solving the Cauchy-Riemann equations 84

Lecture 2. Complex varieties in real hypersurfaces 87
1. Degenerate critical points of smooth functions 87
2. Hermitian symmetry and polarization 89
3. Holomorphic decomposition 90
4. Real analytic hypersurfaces and subvarieties 98
5. Complex varieties, local algebra, and multiplicities 99

Lecture 3. Pseudoconvexity, the Levi form, and points of finite type 105
1. Euclidean convexity 105
2. The Levi form 107
3. Higher order commutators 111
4. Points of finite type 113
5. Commutative algebra 116
6. A return to finite type 121
7. The set of finite type points is open 126

Lecture 4. Kohn's algorithm for subelliptic multipliers 129
1. Introduction 129
2. Subelliptic estimates 130
3. Kohn's algorithm 133
4. Kohn's algorithm for holomorphic and formal germs 134
5. Failure of effectiveness for Kohn's algorithm 139
6. Triangular systems 140
7. Additional remarks 144

Lecture 5. Connections with partial differential equations 147
1. Finite type conditions 147
2. Local regularity for $\overline{\partial}$ 149
3. Hypoellipticity, global regularity, and compactness 150
4. An introduction to L^2-estimates 152

Lecture 6. Positivity conditions 157
1. Introduction 157
2. The classes $\mathcal{P}_k$ 158
3. Intermediate conditions 159
4. The global Cauchy-Schwarz inequality 161
5. A complicated example 164
6. Stabilization in the bihomogeneous polynomial case 166
7. Squared norms and proper mappings between balls 171
8. Holomorphic line bundles 173

Lecture 7. Some open problems 175

Bibliography 177

Dror Varolin
Three Variations on a Theme in Complex Analytic Geometry 183

Lecture 0. Basic notions in complex geometry 189
1. Complex manifolds 189
2. Connections 199
3. Curvature 207
4. Holomorphic line bundles 210

Lecture 1. The Hörmander theorem 217
1. Functional analysis 218
2. The Bochner-Kodaira identity 219
3. Manifolds with boundary 226
4. Density of smooth forms in the graph norm 229
5. Hörmander's theorem 234
6. Singular Hermitian metrics for line bundles 236
7. Application: Kodaira embedding theorem 239
8. Multiplier ideal sheaves and Nadel's Theorems 242
9. Exercises 248

Lecture 2. The $\mathbf{L^2}$ extension theorem 251
1. $\mathbf{L^2}$ extension 251
2. The deformation invariance of plurigenera 259
3. Pluricanonical extension on projective manifolds 265
4. Exercises 275

Lecture 3. The Skoda division theorem 277
1. Statement of the division theorem 277
2. Proof of the division theorem 278
3. Global generation of multiplier ideal sheaves 286
4. Exercises 290

Bibliography 293

Jean-Pierre Demailly
Structure Theorems for Projective and Kähler Varieties 295

0. Introduction 297

1. Numerically effective and pseudo-effective (1,1) classes 298
1.A. Pseudo-effective line bundles and metrics with minimal singularities 298
1.B. Nef line bundles 300
1.C. Description of the positive cones 302
1.D. The Kawamata-Viehweg vanishing theorem 306
1.E. A uniform global generation property due to Y.T. Siu 308
1.F. Hard Lefschetz theorem with multiplier ideal sheaves 309

2. Holomorphic Morse inequalities 310

3. Approximation of closed positive (1,1)-currents by divisors 312
3.A. Local approximation theorem through Bergman kernels 312
3.B. Global approximation of closed (1,1)-currents on a compact complex manifold 314
3.C. Global approximation by divisors 320
3.D. Singularity exponents and log canonical thresholds 326

4. Subadditivity of multiplier ideals and Fujita's approximate Zariski decomposition theorem 329

5. Numerical characterization of the Kähler cone 334
5.A. Positive classes in intermediate (p, p) bidegrees 334
5.B. Numerically positive classes of type (1,1) 335
5.C. Deformations of compact Kähler manifolds 341

6. Structure of the pseudo-effective cone and mobile intersection theory 343
6.A. Classes of mobile curves and of mobile $(n-1, n-1)$-currents 343
6.B. Zariski decomposition and mobile intersections 346
6.C. The orthogonality estimate 352
6.D. Dual of the pseudo-effective cone 354

7. Super-canonical metrics and abundance 357
7.A. Construction of super-canonical metrics 357
7.B. Invariance of plurigenera and positivity of curvature of super-canonical metrics 363
7.C. Tsuji's strategy for studying abundance 364

8. Siu's analytic approach and Păun's non vanishing theorem 365

Bibliography 367

Mihai Păun
Lecture Notes on Rational Polytopes and Finite Generation 371

0. Introduction 373

1. Basic definitions and notations 374

2. Proof of (i) 376
2.1. The case $\mathrm{nd}(\{K_X + Y_{\tau_0} + A\}) = 0$ 377
2.2. The "x method" for sequences 378
2.3. The induced polytope and its properties 382

3. Proof of (ii) 390
3.1. The first step 393
3.2. Iteration scheme 399

References 402

Mircea Mustaţă
Introduction to Resolution of Singularities 405

Lecture 1. Resolutions and principalizations 409
1.1. The main theorems 409
1.2. Strengthenings of Theorem 1.3 410
1.3. Historical comments 414

Lecture 2. Marked ideals 415
2.1. Marked ideals 415
2.2. Derived ideals 419

Lecture 3. Hypersurfaces of maximal contact and coefficient ideals 423
3.1. Hypersurfaces of maximal contact 423
3.2. The coefficient ideal 424

Lecture 4. Homogenized ideals 431
4.1. Basics of homogenized ideals 431
4.2. Comparing hypersurfaces of maximal contact: formal equivalence 433
4.3. Comparing hypersurfaces of maximal contact: étale equivalence 434

Lecture 5. Proof of principalization 437
5.1. The statements 437
5.2. Part I: the maximal order case 438
5.3. Part II: the general case 442
5.4. Proof of principalization 445

Bibliography 449

Robert Lazarsfeld
A Short Course on Multiplier Ideals 451

Introduction 453

Lecture 1. Construction and examples of multiplier ideals 455
Definition of multiplier ideals 455
Monomial ideals 459
Invariants defined by multiplier ideals 460

Lecture 2. Vanishing theorems for multiplier ideals 463
The Kawamata–Viehweg–Nadel vanishing theorem 463
Singularities of plane curves and projective hypersurfaces 465
Singularities of theta divisors 467
Uniform global generation 468

Lecture 3. Local properties of multiplier ideals 471
Adjoint ideals and the restriction theorem 471
The subadditivity theorem 474
Skoda's theorem 475

Lecture 4. Asymptotic constructions 479
Asymptotic multiplier ideals 479
Variants 482
Étale multiplicativity of plurigenera 484
A comparison theorem for symbolic powers 485

Lecture 5. Extension theorems and deformation invariance of plurigenera 487

Bibliography 493

János Kollár
Exercises in the Birational Geometry of Algebraic Varieties 495

1. Birational classification of algebraic surfaces 497

2. Naive minimal models 498

3. The cone of curves 503

4. Singularities 508

5. Flips 513

6. Minimal models 518

Bibliography 523

Christopher D. Hacon
Higher Dimensional Minimal Model Program for Varieties of Log General Type 525

Introduction 527

Lecture 1. Pl-flips 531

Lecture 2. Multiplier ideal sheaves 535
Asymptotic multiplier ideal sheaves 538
Extending pluricanonical forms 540

Lecture 3. Finite generation of the restricted algebra 545
Rationality of the restricted algebra 545
Proof of (1.10) 546

Lecture 4. The minimal model program with scaling 547

Solutions to the exercises 551

Bibliography 555

Alessio Corti, Paul Hacking, János Kollár, Robert Lazarsfeld, and Mircea Mustaţă
Lectures on Flips and Minimal Models 557

Lecture 1. Extension theorems 561
1.1. Multiplier and adjoint ideals 561
1.2. Proof of the Main Lemma 563

Lecture 2. Existence of flips I 565
2.1. The setup 565
2.2. Adjoint algebras 566
2.3. The Hacon–M^{c}Kernan extension theorem 567
2.4. The restricted algebra as an adjoint algebra 567

Lecture 3. Existence of flips II 571

Lecture 4. Notes on Birkar-Cascini-Hacon-M^{c}Kernan 575
4.1. Comparison of 3 MMP's 576
4.2. MMP with scaling 578
4.3. MMP with scaling near $\lfloor \Delta \rfloor$ 578
4.4. Bending it like BCHM 579
4.5. Finiteness of models 581

Bibliography 583