IAS/Park City Mathematics Series
Volume 18, 2009

Introduction

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The 2009 Graduate Summer School of the Institute for Advanced Study/Park City Mathematics Institute was held in Park City, Utah from June 29 to July 17. The topic was "Arithmetic of L-functions". This volume contains the written versions of the graduate lecture courses.

The theory of special values of L-functions and their connection with arithmetic began with the classical class number formula, which expresses the leading term in the Taylor expansion at $s = 0$ of the zeta function $\zeta_K(s)$ of a number field K in terms of arithmetic invariants of K. Due essentially to Dirichlet for quadratic fields and to Dedekind for arbitrary number fields, this formula says

$$\lim_{s\to 0} \frac{\zeta_K(s)}{s^{r_1+r_2-1}} = -\frac{h_K R_K}{w_K}, \tag{1}$$

where r_1 is the number of real embeddings of K, r_2 is the number of conjugate pairs of complex embeddings of K, w_K is the number of roots of unity in K, h_K is the class number of K, and R_K is the regulator of K, the determinant of an $(r_1 + r_2 - 1) \times (r_1 + r_2 - 1)$ matrix whose entries are logarithms of fundamental units of K.

The general theme of the lectures in this volume is various generalizations of the class number formula (1), with an emphasis on the Birch and Swinnerton-Dyer Conjecture and on Stark's Conjecture.

Beginning in the late 1950's, Birch and Swinnerton-Dyer developed a conjectural analogue of (1) for the L-function of an elliptic curve. For the statement see Conjecture 2.10 of the lectures of Gross in this volume.

Stark formulated the initial version of his conjecture in the 1970's. If K/k is a finite Galois extension of number fields, then the zeta function $\zeta_K(s)$ factors into a product of Artin L-functions

$$\zeta_K(s) = \prod_{\rho} L(s,\rho)^{\dim(\rho)} \tag{2}$$

where ρ runs through irreducible complex representations of $\mathrm{Gal}(K/k)$. Stark's Conjecture can be viewed as a factorization of the right hand side of (1), corresponding to the factorization (2) of the left hand side. For the statement see §2.4 of the lectures of Tate in this volume.

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It is tempting to ask for a single general conjecture that will encompass all imaginable generalizations of (1). Currently the best approach in this direction is the Equivariant Tamagawa Number Conjecture (ETNC) of Burns and Flach, which refines and extends an earlier conjecture of Bloch and Kato and deals with L–functions of motives with coefficients. In this volume, we do not go into detail concerning this conjecture in full generality. However, the lectures of Burns and Kings discuss how the ETNC specializes to Stark's Conjecture and to the Birch and Swinnerton-Dyer Conjecture when the motive in question is an Artin motive or an elliptic curve defined over a global field, respectively.

Here is the organization of this volume in more detail. The lectures of Tate introduce Stark's basic conjecture, and Stark's lecture describes the origin of his conjecture. Popescu's lectures discuss integral and p-adic refinements of the conjecture in the case of abelian extensions, and the lectures of Kolster describe a conjecture due to Coates and Sinnott for values of abelian L-functions at negative integers. The lectures of Burns describe the connection between the ETNC and Stark's Conjecture.

Silverberg's lecture gives an introduction to elliptic curves, and the lectures of Gross give an introduction to the Birch and Swinnerton-Dyer Conjecture for elliptic curves over global fields. The lectures of Ulmer discuss the Birch and Swinnerton-Dyer Conjecture for elliptic curves over function fields. The lectures of Birch and Vatsal describe the application of the theory of complex multiplication to the construction of Heegner points (not covered in this volume is Kolyvagin's use of Heegner points to attack the Birch and Swinnerton-Dyer Conjecture). The lectures of Kings describe the connection between the ETNC and the Birch and Swinnerton-Dyer Conjecture.

Rohrlich's lectures discuss root numbers of L-functions. Thanks to the functional equation, in certain cases these root numbers determine the parity of the order of vanishing of an L-function, and thereby have arithmetic implications. Rubin's lectures describe a general theory of Euler systems and Kolyvagin systems, and their application to proving results relating L-functions and arithmetic.

Finally, we would like to express our thanks to all the lecturers for their lectures and their written contributions to this volume. Making this material accessible to a graduate student audience is not easy, and we appreciate their efforts. Without their hard work, this program would not have been possible. We would also like to thank the teaching assistants and the participants, all of whom contributed in essential ways to the success of the program.

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