

Contents

Preface	xiii
Cristian D. Popescu, Karl Rubin, and Alice Silverberg	
Introduction	1
Part I: Stark’s Conjecture	5
John Tate	
Stark’s Basic Conjecture	7
1. L -functions.	9
1.1. Ideal class characters.	10
1.2. Classical abelian L -functions.	10
1.3. Representations of finite groups.	12
1.4. Decomposition, inertia, Frobenius.	15
1.5. Artin L -functions.	15
1.6. Reciprocity and the relation between the two kinds of L -functions.	17
1.7. Brauer’s theorem, meromorphicity.	18
1.8. Functional equation.	19
2. Basic Stark Conjecture.	20
2.1. Class number formula.	20
2.2. S -imprimitive L -functions.	21
2.3. Stark regulator.	22
2.4. Stark’s Basic Conjecture.	23
2.5. The case $r(\chi) = 0$.	27
2.6. The case $r(\chi) = 1$ and Stark units.	28
Bibliography	31
Harold Stark	
The Origin of the “Stark Conjectures”	33
Introduction	35
1. 1966: Class-number one complex quadratic fields	35
2. 1970: The beginnings of the conjectures	37
3. 1975: The first calculations	39
4. The full “over \mathbb{Z} ” conjecture	40
Appendix 1 Heegner.	41
Appendix 2 An evaluation at $s = 0$.	42
Bibliography	44

Cristian D. Popescu	
Integral and p-adic Refinements of the Abelian Stark Conjecture	45
Introduction	47
1. Algebraic preliminaries	50
1.1. Duals	50
1.2. Evaluation maps	52
1.3. Projective modules	54
1.4. Extension of scalars.	56
2. Stark's Main Conjecture in the abelian case	57
2.1. Notations	57
2.2. The L -functions	58
2.3. Stark's Main Conjecture (the abelian case)	59
2.4. Idempotents	62
3. Integral refinements of Stark's Main Conjecture	64
3.1. The Equivariant Tamagawa Number Conjecture – a brief introduction	65
3.2. The integral conjecture of Rubin-Stark	67
4. The Rubin-Stark conjecture in the case $r = 1$	77
4.1. General considerations	77
4.2. Archimedean v_1	78
4.3. Non-archimedean v_1 – the Brumer-Stark conjecture	80
5. Gross-type (p -adic) refinements of the Rubin-Stark conjecture	91
5.1. The set-up	91
5.2. The relevant graded rings $\mathcal{R}(\Gamma)$ and \mathcal{R}_Γ	92
5.3. The conjecture	93
5.4. Linking values of derivatives of p -adic and global L -functions	95
5.5. Sample evidence	98
Bibliography	99
Manfred Kolster	
Special Values of L-functions at Negative Integers	103
Introduction	105
Lecture 1. Iwasawa theory and cohomology	107
1.1. The classical Main Conjecture	107
1.2. Cohomology	111
Lecture 2. Conjectures and results	115
2.1. The Lichtenbaum Conjecture	115
2.2. The Coates-Sinnott Conjecture	118
Bibliography	123

David Burns	
An Introduction to the Equivariant Tamagawa Number Conjecture: the Relation to Stark's Conjecture	125
Introduction	127
Lecture 1. Determinant modules	129
1. Free modules	129
2. Modules of finite projective dimension	131
3. Yoneda two-extensions	132
Lecture 2. A particular case of the ETNC	137
1. Canonical classes	137
2. Statement of the conjecture	139
3. The relation to Stark's Conjecture	140
Lecture 3. Explicit refinements of Stark's Conjecture	143
1. Congruences for values of L -series	144
2. Higher derivatives of L -series and annihilators of class groups	148
Bibliography	151
Part II: Birch and Swinnerton-Dyer Conjecture	153
Alice Silverberg	
Introduction to Elliptic Curves	155
Introduction	157
1. Definitions	157
2. Group Law	158
3. N -torsion	159
4. Elliptic Curves over \mathbb{C}	160
5. Elliptic Curves over Number Fields	160
6. Elliptic Curves over Finite Fields	162
7. Homomorphisms	162
8. Supersingular and Ordinary	163
9. Mod N representations	164
10. Tate Modules and the Isogeny Theorem	164
11. Reduction of Elliptic Curves	165
12. Conjecture of Birch and Swinnerton-Dyer	165
13. Abelian Varieties	165
Bibliography	167
Benedict H. Gross	
Lectures on the Conjecture of Birch and Swinnerton-Dyer	169
Introduction	171
Lecture 1. The Mordell-Weil Theorem	173
1. Torsion on elliptic curves	173
2. Galois cohomology	174
3. A 2-descent	175
4. Heights	177

Lecture 2. Conjectures on L -functions	179
1. The incomplete L -function	179
2. The L -function of an elliptic curve	180
3. Periods	182
4. The conjecture of Birch and Swinnerton-Dyer	184
5. Examples of elliptic curves over function fields	185
6. Examples of elliptic curves over number fields	187
Lecture 3. Progress to date	189
1. Results over function fields	189
2. Constant curves	190
3. Non-constant curves	192
4. Results over number fields	193
5. Local and global heights	194
Appendices	197
A. Reduction modulo v	197
B. Descent computations	199
C. The Tate module and the isogeny theorem	202
D. L -functions of elliptic curves over function fields	203
Bibliography	207
Douglas Ulmer	
Elliptic Curves over Function Fields	211
Introduction	213
Lecture 0. Background on curves and function fields	215
1. Terminology	215
2. Function fields and curves	215
3. Zeta functions	216
4. Cohomology	217
5. Jacobians	217
6. Tate's theorem on homomorphisms of abelian varieties	219
Lecture 1. Elliptic curves over function fields	221
1. Elliptic curves	221
2. Frobenius	222
3. The Hasse invariant	223
4. Endomorphisms	224
5. The Mordell-Weil-Lang-Néron theorem	224
6. The constant case	225
7. Torsion	226
8. Local invariants	228
9. The L -function	229
10. The basic BSD conjecture	230
11. The Tate-Shafarevich group	230
12. Statements of the main results	231
13. The rest of the course	232

Lecture 2. Surfaces and the Tate conjecture	235
1. Motivation	235
2. Surfaces	235
3. Divisors and the Néron-Severi group	236
4. The Picard scheme	237
5. Intersection numbers and numerical equivalence	237
6. Cycle classes and homological equivalence	238
7. Comparison of equivalence relations on divisors	239
8. Examples	239
9. Tate's conjectures T_1 and T_2	241
10. T_1 and the Brauer group	242
11. The descent property of T_1	244
12. Tate's theorem on products	244
13. Products of curves and DPC	245
Lecture 3. Elliptic curves and elliptic surfaces	247
1. Curves and surfaces	247
2. The bundle ω and the height of \mathcal{E}	250
3. Examples	250
4. \mathcal{E} and the classification of surfaces	252
5. Points and divisors, Shioda-Tate	253
6. L -functions and Zeta-functions	254
7. The Tate-Shafarevich and Brauer groups	255
8. The main classical results	256
9. Domination by a product of curves	257
10. Four monomials	257
11. Berger's construction	258
Lecture 4. Unbounded ranks in towers	261
1. Grothendieck's analysis of L -functions	261
2. The case of an elliptic curve	264
3. Large analytic ranks in towers	265
4. Large algebraic ranks	268
Lecture 5. More applications of products of curves	271
1. More on Berger's construction	271
2. A rank formula	272
3. First examples	273
4. Explicit points	274
5. Another example	275
6. Further developments	275
Bibliography	277
Bryan Birch	
Heegner's Proof	281
Preface	283
1. Preliminaries	284
1.1. Step 1. Fundamental region	284
1.2. Step 2. $j(z)$	284
1.3. Step 3. $\Gamma_0(N)$	284
1.4. Step 4. $X_0(N)$ and $F_N(X, Y)$	285
1.5. Step 5. $X_0(2)$ and $X_0(3)$. Integrality	285

2. Weber	286
2.1. Step 6. Quotation	286
2.2. Step 7. Complex multiplication	287
2.3. Step 8. The class polynomial	287
3. Heegner	288
3.1. Application 1. Heegner points of $X_0(N)$	288
3.2. Application 2. Weber's Zoo	288
3.3. Application 3. Gauss' conjecture	289
3.4. Application 4. Points on an elliptic curve	290
Vinayak Vatsal	
Complex Multiplication: a Concise Introduction	293
1. Introduction	295
2. Preliminaries	296
3. The basic definition	297
4. Localization and class field theory	300
5. The modular equation	300
6. The key ingredient: Kronecker congruence	303
7. Proof of the Kronecker congruence	304
8. The ray class field	306
9. Shimura's adelic formulation	308
10. CM points on modular curves	308
Bibliography	313
Guido Kings	
The Equivariant Tamagawa Number Conjecture and the Birch–Swinerton-Dyer Conjecture	315
Introduction	317
Lecture 1. Motives, cohomology and determinants	319
1. Motives	319
2. Realizations	321
3. Motivic cohomology	323
4. L -functions	324
5. Determinants	326
Lecture 2. The equivariant Tamagawa number conjecture	329
1. Rationality conjecture	329
2. Local unramified cohomology	331
3. Global unramified cohomology	332
4. The equivariant Tamagawa number conjecture	334
5. Unramified cohomology for elliptic curves	336
Lecture 3. The relation to the Birch-Swinerton-Dyer conjecture	339
1. The Tamagawa number conjecture and the Birch-Swinerton-Dyer conjecture	339
2. The Selmer group and $R\Gamma_c(\mathbb{Z}_S, T_p E)$	340
3. Local Tamagawa numbers	343
4. Proof of the Main Theorem	345
5. Appendix: Review of some results on elliptic curves	346
Bibliography	349

Part III: Analytic and Cohomological Methods	351
David E. Rohrlich	
Root Numbers	353
Introduction	355
Lecture 1. Trivial central zeros	357
1. Nonexistence of trivial central zeros for Dirichlet L-functions	358
2. Hecke characters and Hecke L-functions	359
3. A family of Hecke L-functions with trivial central zeros	363
4. An open problem	369
5. Evaluation of the quadratic Gauss sum	371
6. Exercises	374
Lecture 2. Local formulas	377
1. The idele class group	377
2. Idele class characters	378
3. The functional equation	382
4. Quadratic root numbers	384
5. Local root numbers	386
6. An open problem	388
7. Epsilon factors	390
8. Exercises	397
Lecture 3. Motivic L-functions	401
1. Artin representations and Artin L-functions	401
2. The functional equation	405
3. Compatible families	407
4. Premotives	413
5. Uniqueness of the functional equation	414
6. An open problem	415
7. Local factors for Artin L-functions	415
8. Exercises	417
Lecture 4. Local formulas in arbitrary dimension	419
1. The local Weil and Weil-Deligne groups	419
2. From Galois representations to Weil-Deligne representations	423
3. An open problem	427
4. Local factors	428
5. Normalizations of the root number in the literature	433
6. Exercises	433
Lecture 5. The minimalist dichotomy	435
1. Elliptic curves	436
2. The minimalist trichotomy	438
3. Elliptic curves revisited	440
4. An open problem	443
5. Exercises	444
Bibliography	445

Karl Rubin	
Euler Systems and Kolyvagin Systems	449
Introduction	451
Lecture 1. Galois cohomology	453
1.1. G -modules.	453
1.2. Characterization of the cohomology groups.	453
1.3. Continuous cohomology.	455
1.4. Change of group.	455
1.5. Selmer groups.	457
1.6. Kummer theory.	458
1.7. The Brauer group.	460
1.8. Local fields and duality.	460
1.9. Unramified and transverse cohomology groups.	461
1.10. Comparing Selmer groups.	463
Lecture 2. Kolyvagin systems	465
2.1. Varying the Selmer structure.	465
2.2. Kolyvagin systems.	466
2.3. Sheaves and monodromy.	467
2.4. Hypotheses on A and \mathcal{F} .	468
2.5. The core Selmer group.	469
2.6. Stepping along the Selmer graph.	470
2.7. Choosing useful primes.	471
2.8. The stub subsheaf.	472
2.9. Recovering the group structure of the dual Selmer group.	474
Lecture 3. Kolyvagin systems for p -adic representations.	477
3.1. Cohomology groups for p -adic representations.	477
3.2. Kolyvagin systems for \mathcal{A} .	479
3.3. Example: units and ideal class groups.	480
3.4. Example: Tate modules of elliptic curves.	485
Lecture 4. Euler systems	487
4.1. Definition of an Euler system.	487
4.2. Example: the cyclotomic unit Euler system	487
4.3. Euler systems and Kolyvagin systems.	489
Lecture 5. Applications	493
5.1. Cyclotomic units and ideal class groups.	493
5.2. Elliptic curves.	493
5.3. General speculation.	496
Bibliography	499