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Introduction

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Geometric group theory refers to the study of discrete groups using tools from topology, geometry, dynamics and analysis. The field is evolving very rapidly and the present volume provides an introduction to and overview of various topics which have played critical roles in this evolution. Specifically, it contains written versions of the nine graduate lecture courses presented during the summer Park City Mathematics Institute (PCMI), held in Park City, Utah from July 1 to July 21, 2012, as well as a contribution from one of the Clay Lecturers, Martin Bridson.

The Clay lecturers for this program were Martin Bridson, Alex Lubotzky and Bill Thurston. We are grateful to all of them for contributing to the energy and excitement of the program.

This was the last conference in which Bill Thurston participated, and we deeply appreciate having had this opportunity to interact with him. His work changed the way we all think about and do mathematics, and in particular had enormous direct influence on the field of geometric group theory. His visionary conjectures on the structure of 3-manifolds provided motivation for many of the directions the field has taken.

The last of these conjectures to be proved were the virtual Haken and the virtual fibering conjectures, which were resolved in a spectacular manner by Ian Agol in the spring of 2012. Agol gave a series of lectures on his proof during the first week of PCMI, which were supplemented with lectures by Piotr Przytycki and Jason Manning. His proof was the culmination of a new approach to subgroup separability developed by Dani Wise over the past fifteen years. The central geometric objects in this approach are CAT(0) cube complexes, which were studied extensively by Sageev, Haglund, Wise and others. The first lecture series in this volume, by Michah Sageev, provides an introduction to the theory of CAT(0) cube complexes and their applications to geometric group theory and topology. This includes a sketch of Caprace and Sageev's *rank rigidity* theorem (which says that under mild conditions CAT(0) cube complexes are products of irreducible complexes which resemble either single lines or Gromov hyperbolic spaces) as well as an introduction to Wise's theory of special cube complexes.

Agol's proof also employed the notion of Dehn fillings of relatively hyperbolic groups. Given a group G which is hyperbolic with respect to a subgroup P, a finite subset $F \subset P$ and a normal subgroup N of P which avoids F, then the Dehn filling theorem says that P/N embeds in the quotient of G by the subgroup $\langle\langle N \rangle\rangle$ normally generated by N; furthermore, $G/\langle\langle N \rangle\rangle$ is relatively hyperbolic with respect to the image of P/N. The second lecture series, by Vincent Guirardel, develops a *small cancellation theory*, which aims in general to understand the quotient of a group by the subgroup normally generated by a given set of subgroups. Other applications include identifying groups which contain every countable group in some quotient as well as constructing hyperbolic groups with various types of pathological behavior.

The third lecture series, by Pierre-Emmanuel Caprace, gives an introduction to the structure of CAT(0) spaces in general. While discrete group actions by isometries on CAT(0) spaces have played a central role in geometric group theory, the emphasis of these lectures is to study the complete isometry group of a locally compact CAT(0) space. This group has the structure of a locally compact topological group (which is not necessarily discrete), and many structural properties of the underlying space can be derived by combining results on locally compact groups with geometric arguments which are often quite elementary.

The lectures by Misha Kapovich provide a detailed introduction to one of the central problems of geometric group theory, that of determining to what extent a group is determined by its geometric actions. Tools from logic (ultralimits) and analysis (quasiconformal maps) are developed and applied to the question of which groups are quasi-isometrically rigid, i.e. virtually determined by such actions.

The lecture series by Mladen Bestvina focuses on the group $Out(F_n)$ of outer automorphisms of a free group and the geometry of the natural spaces on which it acts. The last few years have seen a surge of activity on this subject using the Lipschitz metric on Outer space. Bestvina's lectures give an introduction to this metric and show how it can be used to prove the classification theorem for elements of $Out(F_n)$. The final lecture introduces related complexes, including the free factor complex and the free splitting complex, which were very recently shown to be Gromov hyperbolic, by Bestvina-Feighn and Handel-Mosher, respectively– a development which opens the way to investigate the large scale geometry of $Out(F_n)$.

Many techniques in geometric group theory derive either directly or indirectly from the classical theory of arithmetic groups. The next two lecture series, by David Witte-Morris and Tshachik Gelander, are introductions to the modern development of this classical subject. Witte-Morris's lectures center around the question of whether an arithmetic group can act faithfully on the circle or the real line. They begin with a concrete study of $SL_3(\mathbb{Z})$, using left-orderability to show that it cannot act by homeomorphisms on the circle. They go on to introduce bounded generation, amenable groups and bounded cohomology, all in the context of this problem. Gelander discusses highlights of the theory of lattices in locally compact groups equipped with a Haar measure, elucidating many ideas behind classical theorems as well as providing an introduction to recent developments. Topics include the geometric structure of locally symmetric spaces and rigidity properties of lattices.

Geometric group theory has borrowed many techniques from the theory of dynamical systems. Amie Wilkinson's lectures introduce some of the basic ideas from this subject, including an introduction to geodesic flows in nonpositive curvature, boundaries of Hadamard spaces, and the dynamics of boundary actions of isometry groups. These are all developed along the course of proving J.-P. Otal's theorem that a negatively curved metric on a closed surface is determined by its marked length spectrum.

The ninth set of lectures, by Emmanuel Breuillard, covers several analytic aspects of geometric group theory, beginning with the notions of random walks on groups, amenability and isoperimetry. The lectures include an introduction to Kazdan's property (T) and Lubotzky's property (τ), as well as the recent theory of approximate groups. The aim is to equip the reader with the knowledge and necessary background to understand recent developments regarding expander graphs and related spectral properties of groups, which are summarized in the last lecture.

Finally, the article of Martin Bridson returns to the subject of cube complexes and right-angled Artin groups. He uses techniques from these theories to solve basic algorithmic problems about finitely-presented subgroups of mapping class groups of closed surfaces, and about nilpotent completions of groups.

We sincerely thank all the lecturers for their beautiful lectures as well as for the write-ups they produced for this volume. Thanks also go to the Park City Mathematics Institute for giving us the opportunity to organize this program. The lectures were attended by over 80 graduate students selected after a very competitive process. We were impressed by the strength of this group and we are very optimistic about the future of the subject.