

Contents

Preface	xiii
Mladen Bestvina, Michah Sageev, Karen Vogtmann	
Introduction	1
Michah Sageev	
CAT(0) Cube Complexes and Groups	7
Introduction	9
Lecture 1. CAT(0) cube complexes and pocsets	11
1. The basics of NPC and CAT(0) complexes	11
2. Hyperplanes	16
3. The pocset structure	18
Lecture 2. Cubulations: from pocsets to CAT(0) cube complexes	21
1. Ultrafilters	21
2. Constructing the complex from a pocset	23
3. Examples of cubulations	25
4. Cocompactness and properness	29
5. Roller duality	31
Lecture 3. Rank rigidity	35
1. Essential cores	36
2. Skewering	37
3. Single skewering	37
4. Flipping	38
5. Double skewering	41
6. Hyperplanes in sectors	41
7. Proving rank rigidity	42
Lecture 4. Special cube complexes	45
1. Subgroup separability	45
2. Warmup - Stallings' proof of Marshall Hall's theorem	45
3. Special cube complexes	47
4. Canonical completion and retraction	48
5. Application: separability of quasiconvex subgroups	50
6. Hyperbolic cube complexes are virtually special	51
Bibliography	53

Vincent Guirardel	
Geometric Small Cancellation	55
Introduction	57
Lecture 1. What is small cancellation about?	59
1. The basic setting	59
2. Applications of small cancellation	59
3. Geometric small cancellation	61
Lecture 2. Applying the small cancellation theorem	65
1. When the theorem does not apply	65
2. Weak proper discontinuity	66
3. SQ-universality	68
4. Dehn fillings	69
Lecture 3. Rotating families	71
1. Road-map of the proof of the small cancellation theorem	71
2. Definitions	71
3. Statements	72
4. Proof of Theorem 3.4	73
5. Hyperbolicity of the quotient	76
6. Exercises	78
Lecture 4. The cone-off	79
1. Presentation	79
2. The hyperbolic cone of a graph	81
3. Cone-off of a space over a family of subspaces	83
Bibliography	89
Pierre-Emmanuel Caprace	
Lectures on Proper CAT(0) Spaces and Their Isometry Groups	91
Introduction	93
Lecture 1. Leading examples	95
1. The basics	95
2. The Cartan–Hadamard theorem	96
3. Proper cocompact spaces	97
4. Symmetric spaces	98
5. Euclidean buildings	99
6. Rigidity	100
7. Exercises	101
Lecture 2. Geometric density	103
1. A geometric relative of Zariski density	103
2. The visual boundary	103
3. Convexity	105
4. A product decomposition theorem	106
5. Geometric density of normal subgroups	107
6. Exercises	108

Lecture 3. The full isometry group	111
1. Locally compact groups	111
2. The isometry group of an irreducible space	111
3. de Rham decomposition	113
4. Exercises	115
Lecture 4. Lattices	117
1. Geometric Borel density	117
2. Fixed points at infinity	118
3. Boundary points with a cocompact stabiliser	119
4. Back to rigidity	120
5. Flats and free abelian subgroups	121
6. Exercises	122
Bibliography	123
Michael Kapovich	
Lectures on Quasi-Isometric Rigidity	127
Introduction: What is Geometric Group Theory?	129
Lecture 1. Groups and spaces	131
1. Cayley graphs and other metric spaces	131
2. Quasi-isometries	133
3. Virtual isomorphisms and QI rigidity problem	136
4. Examples and non-examples of QI rigidity	137
Lecture 2. Ultralimits and Morse lemma	141
1. Ultralimits of sequences in topological spaces	141
2. Ultralimits of sequences of metric spaces	142
3. Ultralimits and CAT(0) metric spaces	142
4. Asymptotic cones	143
5. Quasi-isometries and asymptotic cones	144
6. Morse lemma	145
Lecture 3. Boundary extension and quasi-conformal maps	147
1. Boundary extension of QI maps of hyperbolic spaces	147
2. Quasi-actions	148
3. Conical limit points of quasi-actions	149
4. Quasiconformality of the boundary extension	150
Lecture 4. Quasiconformal groups and Tukia's rigidity theorem	157
1. Quasiconformal groups	157
2. Invariant measurable conformal structure for qc groups	158
3. Proof of Tukia's theorem	160
4. QI rigidity for surface groups	162
Appendix.	165
1. Hyperbolic space	165
2. Least volume ellipsoids	166
3. Different measures of quasiconformality	168
Bibliography	171

Mladen Bestvina	
Geometry of Outer Space	173
Introduction	175
Lecture 1. Outer space and its topology	177
1.1. Markings	177
1.2. Metric	178
1.3. Lengths of loops	178
1.4. \mathbb{F}_n -trees	178
1.5. Topology and Action	178
1.6. Thick part and spine	179
1.7. Action of $Out(\mathbb{F}_n)$	179
1.8. Rank 2 picture	181
1.9. Contractibility	181
1.10. Group theoretic consequences	183
Lecture 2. Lipschitz metric, train tracks	185
2.1. Definitions	185
2.2. Elementary facts	185
2.3. Example	186
2.4. Tension graph, train track structure	187
2.5. Folding paths	189
Lecture 3. Classification of automorphisms	193
3.1. Elliptic automorphisms	193
3.2. Hyperbolic automorphisms	193
3.3. Parabolic automorphisms	196
3.4. Reducible automorphisms	197
3.5. Growth	197
3.6. Pathologies	198
Lecture 4. Hyperbolic features	199
4.1. Complex of free factors \mathcal{F}_n	201
4.2. The complex \mathcal{S}_n of free factorizations	201
4.3. Coarse projections	201
4.4. Idea of the proof of hyperbolicity	202
Bibliography	205
Dave Witte Morris	
Some Arithmetic Groups that Do Not Act on the Circle	207
Abstract	209
Lecture 1. Left-orderable groups and a proof for $SL(3, \mathbb{Z})$	211
1A. Introduction	211
1B. Examples	212
1C. The main conjecture	213
1D. Left-invariant total orders	214
1E. $SL(3, \mathbb{Z})$ does not act on the line	215
1F. Comments on other arithmetic groups	217

Lecture 2. Bounded generation and a proof for $\mathrm{SL}(2, \mathbb{Z}[\alpha])$	219
2A. What is bounded generation?	219
2B. Bounded generation of $\mathrm{SL}(2, \mathbb{Z}[\alpha])$	221
2C. Bounded orbits and a proof for $\mathrm{SL}(2, \mathbb{Z}[\alpha])$	223
2D. Implications for other arithmetic groups of higher rank	225
Lecture 3. What is an amenable group?	227
3A. Ponzi schemes	227
3B. Almost-invariant subsets	228
3C. Average values and invariant measures	229
3D. Examples of amenable groups	231
3E. Applications to actions on the circle	232
Lecture 4. Introduction to bounded cohomology	235
4A. Definition	235
4B. Application to actions on the circle	237
4C. Computing $H_b^2(\Gamma; \mathbb{R})$	238
Appendix. Hints for the exercises	241
Bibliography	247
Tsachik Gelander	
Lectures on Lattices and Locally Symmetric Spaces	249
Introduction	251
Lecture 1. A brief overview on the theory of lattices	253
1. Few definitions and examples	253
2. Lattices resemble their ambient group in many ways	254
3. Some basic properties of lattices	254
4. A theorem of Mostow about lattices in solvable groups	256
5. Existence of lattices	258
6. Arithmeticity	259
Lecture 2. On the Jordan–Zassenhaus–Kazhdan–Margulis theorem	261
1. Zassenhaus neighborhood	261
2. Jordan’s theorem	262
3. Approximations by finite transitive spaces	262
4. Margulis’ lemma	263
5. Crystallographic manifolds	263
Lecture 3. On the geometry of locally symmetric spaces and some finiteness theorems	265
1. Hyperbolic spaces	265
2. The thick–thin decomposition	266
3. Presentations of torsion free lattices	267
4. General symmetric spaces	268
5. Number of generators of lattices	269

Lecture 4. Rigidity and applications	273
1. Local rigidity	273
2. Wang's finiteness theorem	274
3. Mostow's rigidity theorem	276
4. Superrigidity and arithmeticity	276
5. Invariant random subgroups and the Nevo–Stuck–Zimmer theorem	277
Bibliography	281
Amie Wilkinson	
Lectures on Marked Length Spectrum Rigidity	283
Introduction	285
Lecture 1. Preliminaries	287
1. Background on negatively curved surfaces	287
2. A key example	288
3. Geodesics in negative curvature	289
4. The geodesic flow	291
Lecture 2. Geometry and dynamics in negative curvature	293
1. Busemann functions and horospheres	293
2. The space of geodesics and the boundary at infinity	297
3. The Liouville current, the cross ratio and the canonical contact form	302
4. Summary: a dictionary	304
Lecture 3. The proof, Part I: A volume preserving conjugacy	305
1. Otal's Proof	308
Lecture 4. The proof, Part II: Volume preserving implies isometry	313
Final Comments	321
Bibliography	323
Emmanuel Breuillard	
Expander Graphs, Property (τ) and Approximate Groups	325
Foreword	327
Lecture 1. Amenability and random walks	329
A. Amenability, Folner criterion	329
B. Isoperimetric inequality, edge expansion	329
C. Invariant means	330
D. Random walks on groups, the spectral radius and Kesten's criterion	331
E. Further facts and questions about growth of groups and random walks	335
F. Exercise: Paradoxical decompositions, Ponzi schemes and Tarski numbers	336

Lecture 2. The Tits alternative and Kazhdan's property (T)	339
A. The Tits alternative	339
B. Kazhdan's property (T)	341
C. Uniformity issues in the Tits alternative, non-amenability and Kazhdan's property (T)	345
Lecture 3. Property (τ) and expanders	347
A. Expander graphs	347
B. Property (τ)	351
Lecture 4. Approximate groups and the Bourgain-Gamburd method	355
A. Which finite groups can be turned into expanders?	355
B. The Bourgain-Gamburd method	357
C. Approximate groups	360
D. Random generators and the uniformity conjecture	362
E. Super-strong approximation	363
Appendix. The Brooks-Burger transfer	365
Bibliography	373
Martin R. Bridson	
Cube Complexes, Subgroups of Mapping Class Groups, and Nilpotent Genus	379
1. Introduction	381
2. Subgroups of mapping class groups	382
3. Fibre products and subdirect products of free groups	385
4. A new level of complication	386
5. The nilpotent genus of a group	387
6. Cubes, RAAGs and $CAT(0)$	389
7. Rips, fibre products and 1-2-3	392
8. Examples template	394
9. Proofs from the template	395
10. The isomorphism problem for subgroups of RAAGs and $\text{Mod}(S)$	396
11. Dehn functions	396
Bibliography	397