About These Study Guides

The Mathematical Association of America's American Mathematics Competitions' website, `www.maa.org/math-competitions`, announces loud and clear:

> Teachers and schools can benefit from the chance to challenge students with interesting mathematical questions that are aligned with curriculum standards at all levels of difficulty.

For over six decades the dedicated and clever folks of the MAA have been creating and collating marvelous, stand-alone mathematical tidbits and sharing them with the world of students and teachers through mathematics competitions. Each question serves as a portal for deep mathematical mulling and exploration. Each is an invitation to revel in the mathematical experience.

And more! In bringing together all the questions that link to one topic, a coherent mathematical landscape, ripe for a guided journey of study, emerges. The goal of this series is to showcase the landscapes that lie within the MAA's competition resources and to invite students, teachers, and all life-long learners, to engage in the mathematical explorations they invite. Learners will not only deepen their understanding of curriculum topics but also gain the confidence to play with ideas and work to become agile intellectual thinkers.

I was recently asked by some fellow mathematics educators what my greatest wish is for our next generation of students. I responded:

> …a personal sense of curiosity coupled with the confidence to wonder, explore, try, get it wrong, flail, go on tangents, make connections, be flummoxed, try some more, wait for epiphanies, lay groundwork for epiphanies, go down false leads, find moments of success nonetheless, savor the "ahas", revel in success, and yearn for more.

Our complex society demands of our next generation not only mastery of quantitative skills, but also the confidence to ask new questions, to innovate, and to succeed. Innovation comes only from bending and pushing ideas and being willing to flail. One must rely on one's wits and on one's common sense. And one must persevere. Relying on memorized answers to previously asked—and answered!—questions does not push the frontiers of business research and science research.

The MAA competition resources provide today's mathematics thinkers, teachers, and doers

- the opportunity to learn and to teach problem solving and
- the opportunity to review the curriculum from the perspective of understanding and clever thinking, letting go of memorization and rote doing.

Each of these study guides

- runs through the entire standard curriculum content of a particular mathematics topic from a sophisticated and mathematically honest point of view,
- illustrates in concrete ways how to implement problem-solving strategies for problems related to the particular mathematics topic, and
- provides a slew of practice problems from the MAA competition resources along with their solutions.

As such, these guides invite you to

- review and deeply understand mathematics topics,
- practice problem solving,
- gain incredible intellectual confidence,

and, above all,

- to enjoy mathematics!

This Guide and Mathematics Competitions

Whether you enjoy the competition experience and are motivated and delighted by it or you, like me, tend to shy away from it, this guide is for you!

We all have our different styles and proclivities for mathematics thinking, doing, and sharing, and they are all good. The point, in the end, lies with the enjoyment of the mathematics itself. Whether you like to solve problems under the time pressure of a clock or while mulling on a stroll, problem solving is a valuable art that will serve you well in all aspects of life.

This guide teaches how to think about content and how to solve challenges. It serves both the competition doers and the competition nondoers. That is, it serves the budding and growing mathematicians we all are.

On Competition Names

This guide pulls together problems from the history of the MAA's American competition resources.

The competitions began in 1950 with the Metropolitan New York Section of the MAA offering a "Mathematical Contest" each year for regional high school students. The competitions became national endeavors in 1957 and adopted the name "Annual High School Mathematics Examination" in 1959. This was changed to the "American High School Mathematics Examination" in 1983.

In these guides, the code "#22, AHSME, 1972", for example, refers to problem number 22 from the 1972 AHSME, Annual/American High School Mathematics Examination.

In 1985 a contest for middle school students was created, the "American Junior High School Mathematics Examination", and shortly thereafter the contests collectively became known as the "American Mathematics Competitions", the AMC for short. In the year 2000 competitions limited to high school students in grades 10 and below were created and the different levels of competitions were renamed the AMC 8, the AMC 10, and the AMC 12.

In these guides, "#13, AMC 12, 2000", for instance, refers to problem number 13 from the 2000 AMC 12 examination.

In 2002, and ever since, two versions of the AMC 10 and the AMC 12 have been administered, about two weeks apart, and these are referred to as the AMC 10A, AMC 10B, AMC 12A, and AMC 12B.

In these guides, "#24, AMC 10A, 2013", for instance, refers to problem number 24 from the 2013 AMC 10A examination.

On Competition Success

Let's be clear:

> "I am using this guide for competition practice. Does this guide promise me 100% success on all mathematics competitions, each and every time?"

Of course not! But this guide does offer, if worked through with care,

- feelings of increased confidence when taking part in competitions,
- clear improvement on how you might handle competition problems,
- clear improvement on how you might handle your emotional reactions to particularly outlandish-looking competition problems.

Mathematics is an intensely human enterprise and one cannot underestimate the effect of emotions when doing mathematics and attempting to solve challenges. This guide gives the human story that lies behind the mathematics content and discusses the human reactions to problem solving.

As we shall learn, the first and the most important effective step in solving a posed problem is:

Step 1: Read the question, have an emotional reaction to it, take a deep breath, and then reread the question.

This guide provides practical content knowledge, problem-solving tools and techniques, and concrete discussion on getting over the barriers of emotional blocks. Even though its goal is not necessarily to improve competition scores, these are the tools that nonetheless lead to that outcome!

This Guide and the Craft of Solving Problems

Success in mathematics—however you wish to define it—comes from a strong sense of self-confidence: the confidence to acknowledge one's emotions and to calm them down, the confidence to pause over ideas and come to educated guesses or conclusions, the confidence to rely on one's wits to navigate through unfamiliar terrain, the confidence to choose understanding over impulsive rote doing, and the confidence to persevere.

Success and joy in science, business, and life doesn't come from programmed responses to preset situations. It comes from agile and adaptive thinking coupled with reflection, assessment, and further adaptation.

Students—and adults too—are often under the impression that one should simply be able to leap into a mathematics challenge and make instant progress of some kind. This is not how mathematics works! It is okay to fumble and flail and to try out ideas that turn out not to help in the end. In fact, this *is* the problem-solving process and making multiple false starts should not at all be dismissed! (Think of how we solve problems in everyday life.)

It is also a natural part of the problem-solving process to react to a problem.

> "This looks scary."
>
> "This looks fun."
>
> "I don't have a clue what the question is even asking!"
>
> "Wow. Weird! Could that really be true?"
>
> "Who cares?"
>
> "I don't get it."
>
> "Is this too easy? I am suspicious."

We are all human, and the first step to solving a problem is to come to terms with our emotional reaction to it—especially if that reaction is one of being overwhelmed. Step 1 to problem solving mentioned earlier is vital.

Once we have our nerves in check, at least to some degree, there are a number of techniques we could try in order to make some progress with the problem.

The ten strategies we briefly outline in Appendix I are discussed in full detail on the MAA's *Curriculum Inspirations* webpage, `www.maa.org/ci`. There you will find essays and videos explaining each technique in full, with worked out examples and slews of further practice examples and their solutions.

This guide also contains worked out examples. Look for the Featured Problems in select chapters where I share with you my own personal thoughts, emotions, and eventual approach in solving a given problem using one of the ten problem-solving strategies.

12
Fitting Formulas to Data Points

Last time I checked my savings account I had a balance of \$275. Checking just now I see that my balance has since grown to \$825. Woohoo!

There are two ways to think about this growth. Additive thinking has me say that I gained \$550 over this recent period of time: $825 = 275 + 550$. Multiplicative thinking has me say that I tripled my money: $825 = 275 \times 3$.

Let's call the time period between the two times I checked my balance one unit of time. If I believe these growth rates are constant, then additive thinking predicts I will have $275 + 550 + 550 = \$1375$ after another unit of time, and multiplicative thinking predicts a balance of $275 \times 3 \times 3 = \$2475$. (Let's hope it is multiplicative growth!) In general, after t units of time, additive thinking predicts a balance of $A(t) = 275 + 550t$ dollars and multiplicative thinking predicts a balance of $M(t) = 275 \times 3^t$ dollars.

We have just found two equations that fit the two data points $(0, 275)$ and $(1, 825)$. The first equation is a *linear model* and the second an *exponential model*. It is a standard part of the school curriculum to have students find linear and exponential equations that fit two given data points.

But what is the best way to do this work if the two data points are not as friendly? How might one go about finding a linear expression $a + bt$ and then an exponential expression $a \cdot b^t$ that fits the data $(3, 201)$ and $(17, 640)$? See Figure 84.

t	$a + bt$
3	201
17	640

t	$a \cdot b^t$
3	201
17	640

FIGURE 84

Exponential Fit

Let's do the supposedly harder one first, the hard way, with a brute force approach. (We'll learn from this process and see how we can later avoid all the hard work.)

We seek an exponential equation

$$M(t) = a \cdot b^t$$

with $M(3) = 201$ and $M(17) = 640$. We have two equations to work with:

$$201 = a \cdot b^3,$$
$$640 = a \cdot b^{17}.$$

Dividing the equations gives $b^{14} = \frac{640}{201}$ and so $b = \left(\frac{640}{201}\right)^{\frac{1}{14}}$. Now substitute this value into the first equation to see that

$$a = \frac{201}{b^3} = 201\left(\frac{201}{640}\right)^{\frac{3}{14}}.$$

Thus

$$M(t) = 201\left(\frac{201}{640}\right)^{\frac{3}{14}}\left(\frac{640}{201}\right)^{\frac{t}{14}}$$

is a scary looking exponential formula that does the trick. (We can simplify it a tad, I suppose.)

Let's now reflect on what we did.

What makes this problem hard is the numbers. The growth is occurring over a period of time that starts at $t = 3$ and ends at $t = 17$, a span of fourteen units of time.

Life would be so much easier if the time period was one unit of time starting at $t = 0$. See Figure 85.

t	$a \cdot b^t$
0	201
1	640

FIGURE 85

Although the growth rate is a bit unfriendly (the data has grown by a factor of $\frac{640}{201}$), it is conceptually straightforward to write an exponential equation that fits this data. We see that the following works (put in $t = 0$ and $t = 1$ to check):

$$N(t) = 201\left(\frac{640}{201}\right)^t.$$

But the real data isn't growing over a period of one unit of time: it grows over fourteen units of time. So we need to slow this formula down by a factor of 14. See Figure 86.

t	$a \cdot b^t$
0	201
14	640

FIGURE 86

How can we do this? How can we make fourteen units of time behave, in some sense, like one unit of time? How do we make $t = 0$ and $t = 14$ behave like $t = 0$ and $t = 1$?

Some mulling and toying suggests replacing t by $\frac{t}{14}$. So let's set

$$W(t) = 201\left(\frac{640}{201}\right)^{\frac{t}{14}}.$$

And putting $t = 0$ and $t = 14$ into this formula shows it is correct.

But the data we were given doesn't start at $t = 0$. It follows a span of fourteen units of time starting at $t = 3$. See Figure 87.

t	$a \cdot b^t$
3	201
17	640

FIGURE 87

Can we adjust the expression we currently have so that $t = 3$ and $t = 17$ now behave like $t = 0$ and $t = 14$?

Some more mulling and toying suggests replacing t by $t - 3$. So set

$$M(t) = 201\left(\frac{640}{201}\right)^{\frac{t-3}{14}}.$$

And inserting $t = 3$ and $t = 17$ as a check shows that this is correct.

That does it. We have a lovely exponential formula that matches the two given data points.

Challenge. Show that our formula here agrees with the brute-force answer we obtained earlier.

Although this approach seems longer and more involved, it teaches a mathematical practice: work hard to avoid hard work! We started with the simplest version of the problem and built up ideas from there.

Just Write Down the Answer!

Now that we have seen the structure of the equations, we have confidence to simply write down fitting formulas, without a lick of scratch work.

Consider, for example, the two data points $(87, 123)$ and $(1000, 14)$. Can you just see that

$$M(t) = 123\left(\frac{14}{123}\right)^{\frac{t-87}{913}}$$

fits? (This equation represents exponential decay.)

Exercise. (a) Write down an exponential equation that fits the data $(10, 10)$ and $(20, 20)$.

(b) Write down an exponential equation that fits the data $(3, 46)$ and $(15, 46)$, if you can.

(c) Write down an exponential equation that fits the data $(8, 11)$ and $(18, 0)$, if you can.

Solution. (a) $P(t) = 10 \cdot (2)^{\frac{t-10}{10}}$.

(b) $Q(t) = 46 \cdot 1^{\frac{t-3}{12}} = 46$.

(c) There is no exponential function of the form $f(t) = a \cdot b^t$ that fits this data. □

Linear Fit

Let's repeat our "avoid the hard work" approach with additive thinking. Let's write down a linear equation for each of the three data tables in Figure 88.

t	$a+bt$
0	201
1	640

t	$a+bt$
0	201
14	640

t	$a+bt$
3	201
17	640

FIGURE 88

For the first table we can use $B(t) = 201 + 439t$.

Slowing down by a factor of 14 gives $C(t) = 201 + 439 \cdot \frac{t}{14}$ for the second table of values.

Making $t = 3$ behave like $t = 0$ gives $A(t) = 201 + 439 \cdot \frac{t-3}{14}$, which is a linear equation fitting the third data table.

Comment. The standard curriculum approach is to use the brute-force method, but with one piece of sophistication. A line connecting the two data points will have slope $\frac{640-201}{17-13} = \frac{439}{14}$, and so a linear equation that fits the data will have the form $A(t) = \frac{439}{14}t + b$. We can now substitute in one data point to solve for b.

Exercise. Consider the following data:

x	y
3	4.2
6	1.8

(a) Write down an exponential equation $y = a \cdot b^x$ that fits this data.

(b) Draw a table of x and $\log y$ values. Find a linear equation that fits the x and $\log y$ data.

(c) Do your equations in parts (a) and (b) match?

(d) Find an equation of the form $y = a \cdot x^b$ that fits the two original data values. (Apply a logarithm first?)

(e) Is there an equation of the form $y = x^a + b$ that fits the original data? (This is a yes/no question.)

Solution. (a) $y = 4.2 \cdot \left(\frac{1}{3}\right)^{\frac{x-1}{5}}$.

(b) We get $\log y = -0.074x + 0.697$ (with coefficients rounded to three decimal places).

(c) The equation $y = 4.2 \cdot 3^{\frac{1-x}{5}}$ can be rewritten as

$$y = 12.6 \cdot (3^{-\frac{1}{3}})^x = 4.2 \cdot 3^{\frac{1}{5}} \cdot (3^{-\frac{1}{5}})^x \approx 5.232 \cdot 0.803^x.$$

The equation $\log y = -0.074x + 0.697$ can be rewritten as

$$y = 10^{0.697} \cdot (10^{-0.074})^x \approx 4.977 \cdot 0.843^x.$$

Are these close to being the same? With less rounding do we get a closer match? (In theory, should the two equations match precisely?)

(d) The equation $y = a \cdot x^b$ can be rewritten as $\log y = b \log x + \log a$. This is an equation of a line for data values $(\log x, \log y)$. The line though $(0, 0.623)$ and $(0.778, 0.255)$ is $\log y = -0.473 \log x + 0.623$, suggesting choosing $a = 10^{0.623} = 4.2$ and $b = -0.473$. (But there may be a concern about rounding errors to contend with.)

(e) No! (Put $x = 1$, $y = 4.2$ into this equation to see that we will need $b = 3.2$, and then put in $x = 6$, $y = 1.8$ to see there is no possible value to adopt for a.) □

More than Two Data Points

Example. Find a quadratic function that fits the data table of Figure 89.

x	y
2	7
5	10
7	3

FIGURE 89

The best thing to do is to just write down the answer! Here it is:

$$p(x) = 7 \cdot \frac{(x-5)(x-7)}{(-3)(-5)} + 10 \cdot \frac{(x-2)(x-7)}{(3)\cdot(-2)} + 3\frac{(x-2)(x-5)}{(5)\cdot(2)}.$$

If we were to expand this out, we'd see that this is indeed a quadratic function. But, of course, this is not the issue in our minds right now. Where did this formula come from?

To understand this formula, start by putting in the value $x = 2$. Notice that the second and third terms are designed to vanish at $x = 2$ and so we have only to contend with the first term,

$$7 \cdot \frac{(x-5)(x-7)}{(-3)(-5)}.$$

When $x = 2$, the numerator and the denominator match (the denominator was designed to do this) so that this term becomes

$$7 \cdot 1,$$

which has the value 7. So we see now that $p(2) = 7 + 0 + 0 = 7$, as desired.

For the value $x = 5$ only the middle term

$$10 \cdot \frac{(x-2)(x-7)}{(3)\cdot(-2)}$$

survives and has value $10 \cdot \frac{(3)\cdot(-2)}{(3)\cdot(-2)} = 10$ for $x = 5$. So $p(5) = 0 + 10 + 0 = 10$, as desired.

In the same way, for the value $x = 7$ only the third term is non-vanishing and we get $P(7) = 0 + 0 + 3 \cdot \frac{5\cdot 2}{5\cdot 2} = 3$.

Thus the quadratic

$$p(x) = 7 \cdot \frac{(x-5)(x-7)}{(-3)(-5)} + 10 \cdot \frac{(x-2)(x-7)}{(3)\cdot(-2)} + 3\frac{(x-2)(x-5)}{(5)\cdot(2)}$$

does indeed produce the desired outputs for the given inputs. If we desires, we can simplify the expression to see that we have

$$p(x) = -\frac{9}{10}x^2 + \frac{219}{30}x - 4.$$

Despite the visual complication of the formula, we can see that its construction is relatively straightforward:

(1) Write a series of numerators that each vanish at all but one of the desired inputs.
(2) Create denominators that cancel the numerators when a specific input is entered.
(3) Use the desired output values as coefficients.

As another example, here's a quadratic that passes though the points $A = (3, 87)$, $B = (10, \pi)$, and $C = (35, \sqrt{2})$. It is

$$q(x) = 87\frac{(x-10)(x-35)}{(-7)(-32)} + \pi\frac{(x-3)(x-35)}{(7)(-28)} + \sqrt{2}\frac{(x-3)(x-10)}{32 \cdot 22}.$$

There is no need to set up a system of three equations in three unknowns to fit a quadratic function to three data points. (And even simplifying the quadratic expressions that appear, if needed at all, is not as tedious as it first might appear.)

Exercise. Find a quartic that fits the data of Figure 90.

x	y
1	a
2	b
3	c
4	d
5	e

FIGURE 90

Solution. The polynomial

$$h(x) = a\frac{(x-2)(x-3)(x-4)(x-5)}{(-1)(-2)(-3)(-4)} + b\frac{(x-1)(x-3)(x-4)(x-5)}{(1)(-1)(-2)(-3)} + \cdots + e\frac{(x-1)(x-2)(x-3)(x-4)}{(4)(3)(2)(1)}$$

does the trick. □

One can use this approach to write down the equation of lines. For example, a degree 1 polynomial that fits the data $(6, 2)$ and $(8, 5)$ is $y = 2\frac{(x-8)}{(-2)} + 5\frac{(x-6)}{(2)}$.

The work of this section establishes the following:

> A degree n polynomial is uniquely determined by its value on $n + 1$ distinct inputs.

Comment. See the study guide on trigonometry to review how to fit formulas to periodic phenomena. See the footnote on page 78.

Additional Problem

62. (#24, AMC 12A, 2005) Let $P(x) = (x-1)(x-2)(x-3)$. For how many polynomials $Q(x)$ does there exist a polynomial of degree 3 such that $P(Q(x)) = P(x) \cdot R(x)$?

(A) 19
(B) 22
(C) 24
(D) 27
(E) 32