Preface

A Brief History of the Contests

Overview

The contests from which the problems in this book were drawn are direct descendants of one created in 1950 by the Metropolitan New York Section of the Mathematical Association of America (MAA). Originally given only to students in the New York City area, the contests were first offered nationally in 1957 under the sponsorship of the MAA and the Society of Actuaries. The primary objective of the contests was, and still is, to promote the study of mathematics by providing high school students with a positive experience in creative problem solving. A secondary objective of identifying mathematically talented students was introduced in 1972, when about 100 top scorers were invited to participate in the USA Mathematical Olympiad (USAMO), an extremely challenging proof-oriented contest.

The 1950 contest was entitled simply, "Mathematical Contest." The word "Annual" was added to the name in 1953. The solution booklet from 1959 is titled, "Annual H.S. Mathematics Examination." Between 1959 and 1982 the titles of the contest and solution booklets used either "Contest" or "Examination," the latter appearing more frequently in the later years. The words "High School" or the initials H.S. were sometimes, but not always, included in the title. In 1983 the contest officially became the American High School Mathematics Examination (AHSME). In the same year a third contest, the American Invitational Mathematics Examination (AIME), was introduced as a stepping stone between the AHSME and the USAMO. Computational in nature, but requiring more creativity than the AHSME, the AIME was deemed a more reliable vehicle for identifying suitable USAMO participants. Two years later a contest for middle school students was created. The American Junior High School Mathematics Examination (AJHSME) was modeled after the AHSME. Shortly thereafter, the entire program of contests became known as the American Mathematics Competitions (AMC).

For reasons that will be mentioned later, it was decided to create a new contest in 2000, limited to students in grades 10 and below. At that time the

AHSME and AJHSME were renamed the AMC 12 and AMC 8, respectively, and the new contest was called the AMC 10.

There was some discussion within the Committee on American Mathematics Competitions (CAMC) about whether top scorers on the AMC 10 should be invited to take the AIME. The argument on one side was that 10th-graders bright enough to qualify for the AIME should be taking the AMC 12. The opposing argument was that a 10th-grader whose previously undiscovered talent is first revealed by a high AMC 10 score should not be denied the opportunity to take the AIME. It was finally decided that AIME invitations would go to those students who obtained a score of 120 out of 150, or else scored in the top 1%, on the AMC 10. The corresponding criteria for the AMC 12 are a score of 100 or placement in the top 5%.

Since 2002 two versions of both the AMC 10 and the AMC 12 have been created every year and offered about two weeks apart. This has been done for two reasons. The immediate impetus for change was a conflict between the 2002 contest date and state holidays in Illinois and Louisiana, which would have precluded participation by students in both states. The new arrangement also addressed a growing concern among CAMC members about contest security. In the past, contests were administered for official credit at any time during a window of a few days. With the advent of contest-focused web sites, it was recognized that contest answers would become public knowledge within hours. Setting a specific date for each form greatly reduces the probability that a student will learn the answers before taking the contest.

The AMC contests have attracted many new sponsors since the early years. In addition to the MAA and the Society of Actuaries, the list of sponsoring organizations grew to include the high school and two-year college honorary mathematics society Mu Alpha Theta, the National Council of Teachers of Mathematics, and the Casualty Actuarial Society by 1971. There are currently more than 21 Contributors and Sponsors of the contests, which demonstrates the wide range of interest in the program.

Format and Scoring

Like their common ancestor, the AHSME, the AMC 10 and AMC 12 are multiple-choice contests with five answer choices for each question. The time allotted for work was initially 80 minutes, increasing to 90 minutes in 1978 and decreasing to 75 minutes in 2000. Although the highest possible score on the contests has always been 150, there have been numerous changes in the number of questions and the scoring formulas used over the years. The first contests had 50 questions, divided into three sections with increasing levels of difficulty. The number of questions in Parts I, II, and III, was 15 (2 points each), 20 (3 points each), and 15 (4 points each), respectively. In 1958 a 20/20/10 division was used with a 2/3/5 scoring. In 1960 the number of questions was reduced to 40, with a 20/10/10 division and a 3/4/5 scoring. In 1968 the number of questions was further reduced to 35, with a 10/10/10/5 division and a 3/4/5/6 scoring. Beginning in 1956, in an effort to discourage random guessing while still encouraging educated guessing, a penalty for incorrect responses was imposed. A student's score was calculated as $P_R - 0.2P_W$ (changed to $P_R - 0.25P_W$ in 1958), where P_R and P_W represent the total point values of questions with correct and incorrect responses.

Beginning in 1974 the contest was no longer partitioned into sections. The contest consisted of 30 5-point questions with the scoring formula $5R - W$, where R and W represent the *number* of questions with correct and incorrect responses. This formula was changed to $30 + 4R - W$ in 1977. The latter formula is neutral with respect to guessing and precludes the possibility of a negative score, a potentially devastating result for an unlucky student! Renewed concern about random guessing prompted another scoring change in 1986, this time to the formula $5R + 2B$, where B is the number of responses left blank. The number of questions was reduced once more in 2000, resulting in a contest of 25 questions each worth 6 points with the scoring formula $6R + 2B$, altered to $6R + 2.5B$ in 2002. The last change, instituted in part to reduce the probability of ties among award recipients, appears to have discouraged students from attempting the more difficult problems. For that reason, the scoring formula changed again in 2007, to $6R + 1.5B$.

Content of the AMC 10 and AMC 12

The scope of the AMC 12 contest is restricted to topics covered in a standard high school curriculum. This seemingly simple guideline leaves significant room for debate. For example, does it exclude a problem whose solution requires Pick's Theorem or the formula for the area of an ellipse? The list of topics considered acceptable for the contest has also changed with time. Probability did not make its first appearance on the contest until 1970, and trigonometry not until 1972. Although many AMC 12 contestants have studied calculus, the CAMC does not consider it to be a standard high school topic at this time.

During the early years of the AHSME, many problems were closely related to routine exercises that students were likely to have done in their classes. That is less the case today, for two reasons. First, in the process of reducing the number of questions on the contest from the original 50 to the present 25, it was desired to maintain a large number of problems whose solutions require creative insight. Consequently, the number of routine problems on each contest has been greatly reduced. Second, beginning in 1994 students were allowed to use calculators on the AHSME. Calculators, including some models with built-in computer algebra systems, have been permitted on the AMC 8, AMC 10 and AMC 12, although not on the AIME or USAMO. Thus, for example, a problem that asks students only to solve an algebraic equation is not appropriate for a calculator-permitted AMC 12. Many good problems involving trigonometry and logarithms are also rendered trivial by calculators. The problems included in this volume were all on calculator-permitted contests, but beginning in 2008 calculators will not be permitted on either the AMC 10 or the AMC 12 contests.

By the late 1990s it was clear that many students in grades 9 and 10 were unable to find many approachable problems on the AHSME. This situation led directly to the creation of the AMC 10. The scope of that contest is restricted to topics that are typically studied in grades 10 and below. The majority of the problems on the AMC 10 contests involve topics from algebra, geometry, probability, and elementary counting and number theory. Specifically excluded from this contest are problems that involve trigonometry, logarithms, complex numbers, and the concept of functions and properties of their graphs. Problems that cover more advanced topics in geometry are also excluded. For example, those that require Heron's formula for finding the area of a triangle, extensions of the Central Angle Theorem, and problems requiring the Power of a Point Theorem are excluded.

Construction of the Contests

The Process

The period between the end of January and mid-February of each year sees the administration of four AMC contests. The two forms of the AMC 12 contest, referred to as the 12A and 12B, are given about two weeks apart. The two forms must be comparable in terms of difficulty and topical balance, yet not so similar as to give an advantage to students taking the later 12B. The AMC 10A and 10B contests are each given on the same day as their AMC 12 counterparts. This arrangement allows the same question to appear on both contests given on the same day, and in fact there is usually an overlap of 40–50% between corresponding forms of the AMC 10 and AMC 12. Because of those considerations, the content of each contest is inextricably linked to the content of the other three, so that all the AMC 10 and 12 contests must be constructed together.

For the contests given in year n, the process begins early in year $n - 2$, when a call goes out to a panel of about 50 problem posers. By early summer a packet of 150 to 200 problems is sent out to a panel of packet reviewers, many of whom are also problem posers. Each reviewer submits a ballot of favorite problems from the packet and a set of comments, including corrections, alternate solutions, and suggestions for improved wording. A tentative draft of each contest is written in the fall and sent out to a panel of draft reviewers for editing. In January of year $n-1$, a second draft of the contests is sent to members of the AMC 10/12 Subcommittee, who meet to discuss it in February. It is common for many of the problems and solutions to be significantly rewritten at the meeting. In addition, the Subcommittee often reorders problems, shifts them from one contest to another, or replaces them entirely. Following the meeting, another draft of each contest is written. Over the next few months that draft undergoes additional reviews, and the contest galleys are printed at the AMC office and sent out for proofreading. The contests are printed during the summer of year $n-1$. By that time the problem packet for year $n+1$ is being reviewed, and the process continues.

Performance statistics over the last several years indicate that we have been reasonably successful in judging the relative difficulty of the A and B forms of each contest. Average scores on the two forms have usually been within a few points of each other. We have not always been as successful in judging the relative difficulty of individual problems. For example, Problem 20 on the 2005 AMC 12B had a higher rate of correct responses (42.8%) than did Problem 6 (31.3%).

In spite of our best efforts, ambiguities and errors occasionally creep into the final product. Fortunately, we have yet to have a difficulty on the AMC 10 contests, but on the 2002 12A contest, the distracter "$N < 1$" was inadvertently written as "$N > 1$", resulting in two correct answer choices.

The Problems

Of the problems that are submitted in each year's problem packet, fewer than half survive the reviewing and editing process. Each problem that ultimately appears on one of the contests must meet several criteria.

- It must be original. The easier problems are often somewhat similar to textbook exercises. However, if a problem or a nearly identical one has been used in another contest or published elsewhere, it must be rejected.

- The best students should be able to make a reasonable attempt to solve all 25 problems within the 75-minute time frame, so problem statements should be brief, and questions should be simple to state. The higher-numbered problems should be difficult by virtue of the insight required for the solution, not the length of the computations.

- It must be possible to state the problem unambiguously.

- The correct answer must not be easily guessable.

- Whenever possible, the wrong answer choices, or distracters, should reflect answers that result from common errors in thinking.
- If a problem has a solution that uses calculus, it must have an equally easy solution that does not.
- Calculators should not provide students with a significant advantage in solving any problem.

The distracters should trap students who think carelessly, but not those who simply read carelessly. For example, the answer obtained by a student who misreads "$x \geq 0$" as "$x \leq 0$" should usually not be included as a distracter. A number of top students missed Problem 3 on the 2004 AMC 10A (Problem 1 on 2004 AMC 12A) contest because they read the problem hastily, expressed their answers in dollars instead of cents, and found the answer they obtained as one of the choices.

It is often difficult to decide what sort of calculator usage constitutes a "significant advantage". A problem that asks only for the value of

$$\sqrt{2007^2 - 2006 \cdot 2008}$$

is clearly not suitable, but if the value must be calculated as one step in the solution of a more difficult problem, the advantage provided by a calculator may be judged to be insignificant. Solutions by programming raise unique issues. For example, if a problem requires the 2007^{th} term of a sequence, has a student gained a significant advantage by programming a calculator to generate successive terms of the sequence? Here the answer depends on the nature of the problem. In some cases it has been decided that the creativity required to write the program is comparable to that required to solve the problem without a calculator.

Conventions in Stating Problems

Contest problems should be stated precisely, but not at the expense of clarity. Thus, for example, the choice of a point in an interval according to a uniform probability distribution is referred to simply as a choice made "at random", and "divisor" is usually substituted for the more precise "positive integer divisor". It is usually understood that triangles cannot have an area of zero, although the phrase "triangle with positive area" has occasionally appeared in problems involving the selection of three points from a lattice of points.

In stating geometry problems, the AMC 10 and AMC 12 contests bridge a stylistic gap between the AMC 8, where problems often consist of only a diagram and a question, and the AIME, in which students must usually construct their own diagrams from a prose description. Problem statements on the

AMC 12 are nearly always diagram-independent, although diagrams are often provided in order to save time when the construction of the diagram is not deemed to be a significant component of the solution. By contrast, the phrase "as shown" appears more frequently in AMC 10 problem statements that would otherwise become lengthy.

Problems in a Physical Context

Problems can often be made more interesting by putting them into a physical context, but attempts to do so occasionally backfire. The problem packet for the 2003 contests contained a submission that read, "Jay wants to cut his heating costs. He takes four wooden cubes 3 feet on each side, and places each of the four at one of the top corners of his 15 foot by 10 foot by 8 foot office. By what percentage is the airspace in the room reduced?" After a reviewer calculated the probable weight of each cube, the wood was changed to styrofoam, but a second reviewer then pointed out that the resulting dimensions of the office would make it useless as a workspace. In its third incarnation, the submission appeared as Problem 3 on the 2003 12A contest.

Problems arising from real-life situations can be especially interesting, but it is usually difficult to describe the problem-inspiring situation concisely. The idea for Problem 3 on the 2006 AMC 12B came from a radio broadcast of a Pitt-Nebraska football game, in which an announcer said, "After a first half in which 34 points were scored, it's been a scoreless third quarter, so now we go to the fourth quarter with the Panthers still trailing by 14." After a moment's thought, the problem poser realized that he had enough information to determine the score. However, it was decided that the unneeded information about quarters might be confusing to students unfamiliar with American football.

Acknowledgments

This book consists of a collection of problems submitted by a large group of people. Quite often a submitted problem will be edited so that it might actually be unrecognizable to the original author, but it is the germ of the idea for the final problem that is so valuable for those creating the final contests. We have a complete record of the source of the problems since 2003, and are able to determine the originator of most of those on the earlier contests, but we apologize in advance for any names that have been inadvertently omitted. Those who have submitted the problems that make these contests possible include:

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