The Sixty-Second William Lowell Putnam Mathematical Competition—December 1, 2001

Questions Committee: Eugene Luks (Chair), Titu Andreescu, Andrew J. Granville, and Carl Pomerance
See page 55 for hints.

A1.

Consider a set S and a binary operation $*$, that is, for each $a, b \in S$, $a * b \in S$. Assume $(a * b) * a = b$ for all $a, b \in S$. Prove that $a * (b * a) = b$ for all $a, b \in S$.

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A2.

You have coins $C_1, C_2, \ldots, C_n$. For each k, C_k is biased so that, when tossed, it has probability $\frac{1}{2k+1}$ of falling heads. If the n coins are tossed, what is the probability that the number of heads is odd? Express the answer as a rational function of n.

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A3.

For each integer m, consider the polynomial

$$P_m(x) = x^4 - (2m+4)x^2 + (m-2)^2.$$

For what values of m is $P_m(x)$ the product of two nonconstant polynomials with integer coefficients?

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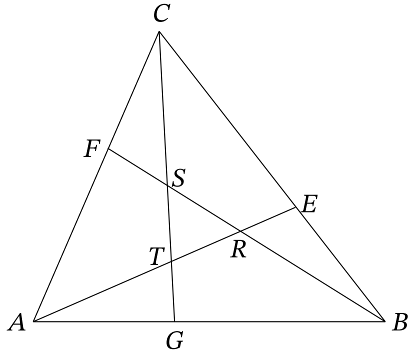
A4.

Triangle ABC has area 1. Points E, F, G lie, respectively, on sides BC, CA, AB such that AE bisects BF at point R, BF bisects CG at point S, and CG bisects AE at point T. Find the area of the triangle RST.



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A5.

Prove that there are unique positive integers a, n such that

$$a^{n+1} - (a+1)^n = 2001.$$

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A6.

Can an arc of a parabola inside a circle of radius 1 have a length greater than 4?

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B1.

Let n be an even positive integer. Write the numbers $1, 2, \ldots, n^2$ in the squares of an $n \times n$ grid so that the kth row, from left to right, is

$$(k-1)n+1, (k-1)n+2, \ldots, (k-1)n+n.$$

Color the squares of the grid so that half of the squares in each row and in each column are red and the other half are black (a checkerboard coloring is one possibility). Prove that for each coloring, the sum of the numbers on the red squares is equal to the sum of the numbers on the black squares.

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B2.

Find all pairs of real numbers (x, y) satisfying the system of equations

$$\begin{aligned} \frac{1}{x} + \frac{1}{2y} &= (x^2 + 3y^2)(3x^2 + y^2) \\ \frac{1}{x} - \frac{1}{2y} &= 2(y^4 - x^4). \end{aligned}$$

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B3.

For any positive integer n, let $\langle n \rangle$ denote the closest integer to $\sqrt{n}$. Evaluate

$$\sum_{n=1}^{\infty} \frac{2^{\langle n \rangle} + 2^{-\langle n \rangle}}{2^n}.$$

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B4.

Let S denote the set of rational numbers different from $-1, 0$, and 1. Define $f : S \to S$ by $f(x) = x - 1/x$. Prove or disprove:

$$\bigcap_{n=1}^{\infty} f^{(n)}(S) = \emptyset,$$

where $f^{(n)} = \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}$.

(Note: $f(S)$ denotes the set of all values $f(s)$ for $s \in S$.)

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B5.

Let a and b be real numbers in the interval $(0, \frac{1}{2})$ and let g be a continuous real-valued function such that $g(g(x)) = ag(x) + bx$ for all real x. Prove that $g(x) = cx$ for some constant c.

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B6.

Assume that $(a_n)_{n\geq 1}$ is an increasing sequence of positive real numbers such that $\lim \frac{a_n}{n} = 0$. Must there exist infinitely many positive integers n such that

$$a_{n-i} + a_{n+i} < 2a_n \text{ for } i = 1, 2, \ldots, n-1?$$

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A1. In $(a * b) * a = b$, replace a with various expressions involving the $*$ operation.

A2. Find a recursive relation for P_n, the desired probability for n coins.

A3. Find the roots of $P_m(x)$ and think about how one or two of these roots could be the roots of a divisor of $P_m(x)$.

A4. Create variables for EC/BC, FA/CA, GB/AB and relate them by computing areas of relevant triangles.

A5. Note that $2001 = a^{n+1} - (a+1)^n \equiv -1 \pmod{a}$. Possible values of a, n can be cut down by similar tricks.

A6. Consider a skinny parabola with vertex on the boundary of the disc and axis of symmetry along its diameter.

B1. Decompose the number on each square into a portion that is constant within each row and a portion that is constant within each column.

B2. Start by considering the sum and the difference of the two given equations.

B3. Group terms based on their value of $\langle n \rangle$.

B4. Let $H(p/q) = |p| + |q|$, where $\gcd(p, q) = 1$. Show that $H(f(x)) > H(x)$ for all x.

B5. *Hint 1 of 2.* Prove that g is bijective. *Hint 2 of 2.* For $x_0 \in \mathbb{R}$ and $n \in \mathbb{Z}$, find a formula for the nth iterate $x_n = g^{(n)}(x_0)$.

B6. Find the value of n for which $a_n - cn$ is maximal for some $c > 0$.

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A1. (189, 1, 0, …, 0, 0, 8, 2) 95.0%

Consider a set S and a binary operation $*$; that is, for each $a, b \in S$, $a * b \in S$. Assume $(a * b) * a = b$ for all $a, b \in S$. Prove that $a * (b * a) = b$ for all $a, b \in S$.

Solution. Replacing a by $b * a$ in the condition implies $((b * a) * b) * (b * a) = b$ for all $a, b \in S$, and hence because $(b * a) * b = a$, it follows that $a * (b * a) = b$ for all $a, b \in S$.

A2. (163, 6, 5, …, 1, 1, 14, 10) 87.0%

You have coins $C_1, C_2, \ldots, C_n$. For each k, C_k is biased so that, when tossed, it has probability $\frac{1}{2k+1}$ of falling heads. If the n coins are tossed, what is the probability that the number of heads is odd? Express the answer as a rational function of n.

Answer. The probability is $n/(2n + 1)$.

Solution 1. Let P_n denote the desired probability. Then $P_1 = 1/3$. For $j > 1$ there are two ways to obtain an odd number of heads when flipping the first j coins: either there were an odd number of heads in the first $j-1$ coins, and the jth coin is a head, or there were an even number of heads in the first $j - 1$ coins, and the jth coin is a tail. Thus for $j > 1$,

$$P_j = \left(\frac{2j}{2j+1}\right) P_{j-1} + \left(\frac{1}{2j+1}\right)(1 - P_{j-1})$$
$$= \left(\frac{2j-1}{2j+1}\right) P_{j-1} + \frac{1}{2j+1}.$$

The recurrence yields $P_2 = 2/5$, $P_3 = 3/7$, and simple induction shows that $P_j = j/(2j + 1)$ for general j.

Solution 2. (Richard Stanley) The following is a noninductive argument. Put $f(x) = \prod_{k=1}^{n}(x+2k)/(2k+1)$. Then the coefficient of x^i in $f(x)$ is the probability of getting exactly i heads. Thus the required probability is $(f(1) - f(-1))/2$, and both values of f can be computed directly: $f(1) = 1$, and

$$f(-1) = \frac{1}{3} \cdot \frac{3}{5} \cdot \cdots \cdot \frac{2n-1}{2n+1} = \frac{1}{2n+1}.$$

A3. (97, 33, 5, …, 24, 22, 7, 12) 67.5%

For each integer m, consider the polynomial

$$P_m(x) = x^4 - (2m + 4)x^2 + (m - 2)^2.$$

For what values of m is $P_m(x)$ the product of two nonconstant polynomials with integer coefficients?

Answer. $P_m(x)$ can be written as such a product if and only if m is either a perfect square or twice a perfect square.

Solution. By the quadratic formula, if $P_m(x) = 0$, then $x^2 = m \pm 2\sqrt{2m} + 2$, and hence the four roots of P_m are given by $S = \{\pm\sqrt{m} \pm \sqrt{2}\}$. If P_m factors into two nonconstant polynomials over the integers, then some one- or two-element subset of S gives the roots of a polynomial with integer coefficients.

If this subset has a single element, say $\sqrt{m} \pm \sqrt{2}$, it is the root of a linear polynomial with integer coefficients, so it must be a rational number. Then $(\sqrt{m} \pm \sqrt{2})^2 = 2 + m \pm 2\sqrt{2m}$ is an integer, so m is twice a perfect square, say $m = 2n^2$. But $\sqrt{m} \pm \sqrt{2} = (n \pm 1)\sqrt{2}$ is only rational if $n = \pm 1$, that is, only if $m = 2$.

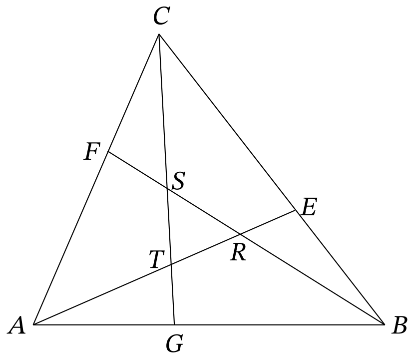
If the subset contains two elements, then it is one of the six subsets $\{\sqrt{m} \pm \sqrt{2}\}$, $\{\sqrt{2} \pm \sqrt{m}\}$, or $\{\pm(\sqrt{m} + \sqrt{2})\}$. In all cases, the sum and the product of the elements of the subset must be rational numbers. In the first case, this means $2\sqrt{m} \in \mathbb{Q}$, so m is a perfect square. In the second case, $2\sqrt{2} \in \mathbb{Q}$, which is a contradiction. In the third case, $(\sqrt{m} + \sqrt{2})^2 \in \mathbb{Q}$, or $m + 2 + 2\sqrt{2m} \in \mathbb{Q}$, which means that m is twice a perfect square.

Therefore $P_m(x)$ factors into two nonconstant polynomials over the integers only if m is either a square or twice a square. However, in either of these cases, one can see directly that $P_m(x)$ does actually factor. If m is a perfect square, then $(x - \sqrt{m} - \sqrt{2})(x - \sqrt{m} + \sqrt{2}) = x^2 - 2\sqrt{m}x + (m - 2)$ is a factor of $P_m(x)$ with integer coefficients. If m is twice a square, then $(x - \sqrt{m} - \sqrt{2})(x + \sqrt{m} + \sqrt{2}) = x^2 - (m + 2\sqrt{2m} + 2)$ is a factor with integer coefficients. This completes the proof.

Reinterpretation. If m is neither a square nor twice a square, then the number fields $\mathbb{Q}(\sqrt{m})$ and $\mathbb{Q}(\sqrt{2})$ are distinct quadratic fields, so their compositum $\mathbb{Q}(\sqrt{m}, \sqrt{2})$ is a number field of degree 4, whose Galois group acts transitively on $\{\pm\sqrt{m} \pm \sqrt{2}\}$. Thus P_m is irreducible.

A4. (29, 7, 8, …, 10, 10, 47, 89) 22.0%

Triangle ABC has area 1. Points E, F, G lie, respectively, on sides BC, CA, AB such that AE bisects BF at point R, BF bisects CG at point S, and CG bisects AE at point T. Find the area of the triangle RST.



Answer. The area of RST is $\frac{7-3\sqrt{5}}{4}$.

Solution. Choose r, s, t so that $EC = rBC, FA = sCA, GB = tAB$, and let $[XYZ]$ denote the area of triangle XYZ. Then $[ABE] = [AFE]$ since the triangles have the same altitude ($FR = BR$) and a common base AE. By the sine formula for the area of a triangle,

$$[ABE] = \frac{1}{2}(AB)(BE)\sin(\angle ABE) = (BE/BC)[ABC] = 1 - r$$

and

$$[ECF] = \frac{1}{2}(EC)(CF)\sin(\angle ECF) = (EC/BC)(CF/CA)[ABC] = r(1-s).$$

Adding this all up yields

$$\begin{aligned} 1 &= [ABE] + [AFE] + [ECF] \\ &= 2(1-r) + r(1-s) = 2 - r - rs \end{aligned}$$

or $r(1+s) = 1$. Similarly, $s(1+t) = t(1+r) = 1$.

Let $f : [0, \infty) \to [0, \infty)$ be the function given by $f(x) = 1/(1+x)$; then $f(f(f(r))) = r$. However, $f(x)$ is strictly decreasing in x, so $f(f(x))$ is increasing and $f(f(f(x)))$ is decreasing. Thus there is at most one x such that $f(f(f(x))) = x$. Since the equation $f(z) = z$ has a positive root $z = (-1+\sqrt{5})/2$, it must be that $r = s = t = z$.

Now compute

$$\begin{aligned} [ABF] &= (AF/AC)[ABC] = z, \\ [ABR] &= (BR/BF)[ABF] = z/2, \\ [BCS] &= [CAT] = z/2, \end{aligned}$$

and

$$[RST] = |[ABC] - [ABR] - [BCS] - [CAT]| = |1 - 3z/2| = \frac{7-3\sqrt{5}}{4}.$$

Remark. The key relation $r(1 + s) = 1$ can also be derived using homogeneous coordinates or vectors.

A5. (12, 9, 10, …, 9, 41, 33, 86) 15.5%

Prove that there are unique positive integers a, n such that

$$a^{n+1} - (a + 1)^n = 2001.$$

Solution. We prove that the unique solution is $a = 13, n = 2$.

Suppose $a^{n+1} - (a+1)^n = 2001$. Since $a^{n+1} - [(a+1)^n - 1]$ is a multiple of a, a divides $2002 = 2 \times 7 \times 11 \times 13$.

Since 2001 is divisible by 3, it must be that $a \equiv 1 \pmod 3$, otherwise one of a^{n+1} and $(a + 1)^n$ is a multiple of 3 and the other is not, so their difference cannot be divisible by 3. Now $a^{n+1} \equiv 1 \pmod 3$, so it must be that $(a + 1)^n \equiv 1 \pmod 3$, which can only happen if n is even, and in particular at least 2.

If a is even, then $a^{n+1} - (a + 1)^n \equiv -(a + 1)^n \pmod 4$. Since n is even, $-(a + 1)^n \equiv -1 \pmod 4$. Since $2001 \equiv 1 \pmod 4$, this is impossible. Thus a is odd, and so must divide $1001 = 7 \times 11 \times 13$. Moreover, $a^{n+1} - (a + 1)^n \equiv a \pmod 4$, so $a \equiv 1 \pmod 4$.

Of the divisors of $7 \times 11 \times 13$, those congruent to 1 modulo 3 are precisely those not divisible by 11 (since 7 and 13 are both congruent to 1 modulo 3). Thus a divides 7×13. Now $a \equiv 1 \pmod 4$ is only possible if a divides 13.

We cannot have $a = 1$, since $1-2^n \neq 2001$ for any n. Thus the only possibility is $a = 13$. The equality $13^3 - 14^2 = 2197 - 196 = 2001$ shows that $a = 13, n = 2$ is a solution; all that remains is to check that no other n works. In fact, if $n > 2$, then $13^{n+1} \equiv 2001 \equiv 1 \pmod 8$. But $13^{n+1} \equiv 13 \pmod 8$ since n is even, giving a contradiction. Thus $a = 13, n = 2$ is the unique solution.

Remark. Once one has that n is even, one can use that $2002 = a^{n+1} + 1 - (a+1)^n$ is divisible by $a + 1$ to rule out cases.

A6. (1, 0, 0, …, 0, 1, 57, 141) 0.5%

Can an arc of a parabola inside a circle of radius 1 have a length greater than 4?

Answer. Yes, it can have length greater than 4.

Solution. Inside the circle $x^2 + (y - 1)^2 = 1$, consider the arc of the parabola $y = Ax^2$, where we initially assume that $A > 1/2$. This intersects the circle in three points, $(0, 0)$ and $(\pm\sqrt{2A - 1}/A, (2A-1)/A)$. We claim that for A sufficiently large, the length L of the parabolic arc between $(0, 0)$ and $(\sqrt{2A - 1}/A, (2A-1)/A)$ is greater than 2, which implies the desired result by symmetry. We express L

using the usual formula for arc length:

$$L = \int_{x_0}^{x_1} \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2}\, dx$$
$$= \int_0^{\sqrt{2A-1}/A} \sqrt{1 + (2Ax)^2}\, dx.$$

Substituting $u = 2Ax$ yields $\frac{1}{2A} \int_0^{2\sqrt{2A-1}} \sqrt{1+u^2}\, du$. Noting that

$$\frac{1}{2A} \int_0^{2\sqrt{2A-1}} u\, du = 2 - \frac{1}{A},$$

we can take u out of the integrand to obtain

$$L = 2 + \frac{1}{2A} \left(\int_0^{2\sqrt{2A-1}} (\sqrt{1+u^2} - u)\, du - 2 \right)$$

Now, for $u \geq 0$,

$$\sqrt{1+u^2} - u = \frac{1}{\sqrt{1+u^2}+u} > \frac{1}{2\sqrt{1+u^2}} \geq \frac{1}{2(u+1)};$$

since $\int_0^\infty du/(2(u+1))$ diverges, so does $\int_0^\infty (\sqrt{1+u^2} - u)\, du$. Hence for sufficiently large A, $\int_0^{2\sqrt{2A-1}} (\sqrt{1+u^2} - u)\, du > 2$ and hence $L > 2$.

Remark. A numerical computation shows that one must take $A > 34.7$ to obtain $L > 2$, and that the maximum value of L is about 4.0027, achieved for $A \approx 94.1$.

B1. (136, 25, 7, ..., 3, 3, 20, 6) 84.0%

Let n be an even positive integer. Write the numbers $1, 2, \ldots, n^2$ in the squares of an $n \times n$ grid so that the kth row, from left to right, is

$$(k-1)n + 1, (k-1)n + 2, \ldots, (k-1)n + n.$$

Color the squares of the grid so that half of the squares in each row and in each column are red and the other half are black (a checkerboard coloring is one possibility). Prove that for each coloring, the sum of the numbers on the red squares is equal to the sum of the numbers on the black squares.

Solution. Let R (resp. B) denote the set of red (resp. black) squares in such a coloring, and for $s \in R \cup B$, let $f(s)n + g(s) + 1$ denote the number written in square s, where $0 \leq f(s), g(s) \leq n - 1$. Then it is clear that the value of $f(s)$

depends only on the row of s, while the value of $g(s)$ depends only on the column of s. Since every row contains exactly $n/2$ elements of R and $n/2$ elements of B,

$$\sum_{s\in R} f(s) = \sum_{s\in B} f(s).$$

Similarly, because every column contains exactly $n/2$ elements of R and $n/2$ elements of B,

$$\sum_{s\in R} g(s) = \sum_{s\in B} g(s).$$

It follows that

$$\sum_{s\in R} f(s)n + g(s) + 1 = \sum_{s\in B} f(s)n + g(s) + 1,$$

as desired.

Remark. Richard Stanley points out a theorem of Ryser (see [**Ryser**, Theorem 3.1]) that can also be applied. Namely, if A and B are 0-1 matrices with the same row and column sums, then there is a sequence of operations on 2×2 matrices of the form

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

or vice versa, which transforms A into B. If 0 and 1 are identified with red and black, respectively, then the given coloring and the checkerboard coloring both satisfy the sum condition. Since the desired result is clearly true for the checkerboard coloring, and performing the matrix operations does not affect this, the desired result follows in general.

B2. (38, 1, 2, …, 2, 7, 52, 98) 20.5%

Find all pairs of real numbers (x, y) **satisfying the system of equations**

$$\frac{1}{x} + \frac{1}{2y} = (x^2 + 3y^2)(3x^2 + y^2),$$
$$\frac{1}{x} - \frac{1}{2y} = 2(y^4 - x^4).$$

Answer. The unique such pair is

$$(x, y) = \left(\frac{3^{1/5}+1}{2}, \frac{3^{1/5}-1}{2}\right).$$

Solution. Adding and subtracting the two given equations yields the equivalent pair of equations

$$2/x = x^4 + 10x^2y^2 + 5y^4,$$
$$1/y = 5x^4 + 10x^2y^2 + y^4.$$

Multiplying the first equation by x and the second by y, then adding and subtracting the two resulting equations, gives another pair of equations equivalent to the given ones:

$$3 = (x + y)^5,$$
$$1 = (x - y)^5.$$

This then leads to the unique solution given above.

B3. (120, 6, 5, . . . , 12, 17, 5, 35) 65.5%

For any positive integer n, let $\langle n \rangle$ denote the closest integer to $\sqrt{n}$. Evaluate

$$\sum_{n=1}^{\infty} \frac{2^{\langle n \rangle} + 2^{-\langle n \rangle}}{2^n}.$$

Answer. The sum is 3.

Solution. Let k be a positive integer. Since $(k - 1/2)^2 = k^2 - k + 1/4$ and $(k + 1/2)^2 = k^2 + k + 1/4$, a positive integer n satisfies $\langle n \rangle = k$ if and only if $k^2 - k + 1 \le n \le k^2 + k$. Hence

$$\sum_{n=1}^{\infty} \frac{2^{\langle n \rangle} + 2^{-\langle n \rangle}}{2^n} = \sum_{k=1}^{\infty} \sum_{\langle n \rangle = k} \frac{2^{\langle n \rangle} + 2^{-\langle n \rangle}}{2^n}$$
$$= \sum_{k=1}^{\infty} \sum_{n=k^2-k+1}^{k^2+k} \frac{2^k + 2^{-k}}{2^n}.$$

Summing the finite geometric series yields

$$= \sum_{k=1}^{\infty} (2^k + 2^{-k})(2^{-k^2+k} - 2^{-k^2-k})$$
$$= \sum_{k=1}^{\infty} (2^{-k(k-2)} - 2^{-k(k+2)})$$
$$= \sum_{k=1}^{\infty} 2^{-k(k-2)} - \sum_{k=3}^{\infty} 2^{-(k-2)k}$$
$$= 2^{-1(-1)} + 2^{-2(0)} = 3,$$

where the index of summation in the second term was shifted by 2.

Reinterpretation. Rewrite the sum as $\sum_{n=1}^{\infty} 2^{-(n+\langle n \rangle)} + \sum_{n=1}^{\infty} 2^{-(n-\langle n \rangle)}$. Note that $\langle n \rangle \ne \langle n+1 \rangle$ if and only if $n = m^2 + m$ for some $m \ge 1$. Thus $n + \langle n \rangle$ and

$n - \langle n \rangle$ each increase by 1 except at $n = m^2 + m$, where the former skips from $m^2 + 2m$ to $m^2 + 2m + 2$ and the latter repeats the value m^2. Thus the sums are

$$\left(\sum_{n=2}^{\infty} 2^{-n} - \sum_{m=1}^{\infty} 2^{-(m+1)^2}\right) + \left(\sum_{n=0}^{\infty} 2^{-n} + \sum_{m=1}^{\infty} 2^{-m^2}\right) = \frac{1}{2} + 2 + \frac{1}{2} = 3,$$

where the first and third summations contribute 1/2 and 2, respectively, and the terms from the second and fourth summations cancel except for one term equal to 1/2.

B4. (43, 21, 28, ..., 3, 0, 34, 71) 46.0%

Let S denote the set of rational numbers different from $-1, 0,$ and 1. Define $f : S \to S$ by $f(x) = x - 1/x$. Prove or disprove:

$$\bigcap_{n=1}^{\infty} f^{(n)}(S) = \emptyset,$$

where $f^{(n)} = \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}$.

(Note: $f(S)$ denotes the set of all values $f(s)$ for $s \in S$.)

Answer. We prove the assertion that the intersection is empty.

Solution. For a rational number p/q expressed in lowest terms, define its *height* $H(p/q)$ to be $|p|+|q|$. Then for any $p/q \in S$ expressed in lowest terms, $H(f(p/q)) = H((p^2 - q^2)/(pq)) = |q^2 - p^2| + |pq|$. This is because any prime common divisor of $p^2 - q^2$ and pq must divide one of $p + q$ or $p - q$ and one of p or q, but by taking appropriate linear combinations of these multiples, we find that it must divide both p and q, which is impossible as p and q were assumed to be relatively prime. Since by assumption p and q are nonzero integers with $|p| \neq |q|$,

$$\begin{aligned} H(f(p/q)) - H(p/q) &= |q^2 - p^2| + |pq| - |p| - |q| \\ &\geq 3 + |pq| - |p| - |q| \\ &= (|p| - 1)(|q| - 1) + 2 \geq 2. \end{aligned}$$

It follows that $f^{(n)}(S)$ consists solely of numbers of height strictly larger than $2n + 2$, and hence

$$\bigcap_{n=1}^{\infty} f^{(n)}(S) = \emptyset.$$

Remark. Many choices for the height function are possible: one can take $H(p/q) = \max\{|p|, |q|\}$, or $H(p/q)$ equal to the total number of prime factors of p and q, and so on. The key properties of the height function are that on one hand, there are only finitely many rationals with height below any finite bound, and on the other hand, the height function is a sufficiently "algebraic" function of its argument that one can relate the heights of p/q and $f(p/q)$.

B5. (2, 0, 0, …, 3, 3, 82, 110) 1.0%

Let a and b be real numbers in the interval $(0, \frac{1}{2})$ and let g be a continuous real-valued function such that $g(g(x)) = ag(x) + bx$ for all real x. Prove that $g(x) = cx$ for some constant c.

Solution. Note that $g(x) = g(y)$ implies that $g(g(x)) = g(g(y))$ and hence $x = y$ from the given equation. That is, g is injective. Since g is also continuous, g is either strictly increasing or strictly decreasing. Moreover, g cannot tend to a finite limit L as $x \to \infty$, or else $g(g(x)) - ag(x) = bx$ with the left side bounded and the right side unbounded. Similarly, g cannot tend to a finite limit as $x \to -\infty$. Together with monotonicity and continuity, this yields that g is also surjective.

Pick x_0 arbitrary, and define x_n for all $n \in \mathbb{Z}$ recursively by $x_{n+1} = g(x_n)$ for $n > 0$, and $x_{n-1} = g^{-1}(x_n)$ for $n < 0$. Let $r_1 = (a + \sqrt{a^2 + 4b})/2$ and $r_2 = (a - \sqrt{a^2 + 4b})/2$ be the roots of $x^2 - ax - b = 0$, so that $r_1 > 0 > r_2$ and $1 > |r_1| > |r_2|$. Since $x_{n+2} = ax_{n+1} + bx_n$ for all $n \in \mathbb{Z}$, by standard facts about recurrence relations, there exist $c_1, c_2 \in \mathbb{R}$ such that $x_n = c_1 r_1^n + c_2 r_2^n$ for all $n \in \mathbb{Z}$.

Suppose g is strictly increasing. If $c_2 \neq 0$ for some choice of x_0, then x_n is dominated by r_2^n for n sufficiently negative. But taking x_n and x_{n+2} for n sufficiently negative of the right parity, we get $0 < x_n < x_{n+2}$, contradicting the fact that $g(x_n) > g(x_{n+2})$. Thus $c_2 = 0$; since $x_0 = c_1$ and $x_1 = c_1 r_1$, $g(x) = r_1 x$ for all x. Analogously, if g is strictly decreasing, then $c_2 = 0$ or else x_n is dominated by r_1^n for n sufficiently positive. But taking x_n and x_{n+2} for n sufficiently positive of the right parity, we get $0 < x_{n+2} < x_n$, contradicting the fact that $g(x_{n+2}) < g(x_n)$. Thus in that case, $g(x) = r_2 x$ for all x.

B6. (9, 5, 2, …, 2, 0, 40, 142) 8.0%

Assume that $(a_n)_{n \geq 1}$ is an increasing sequence of positive real numbers such that $\lim \frac{a_n}{n} = 0$. Must there exist infinitely many positive integers n such that

$$a_{n-i} + a_{n+i} < 2a_n \text{ for } i = 1, 2, \dots, n-1?$$

Answer. Yes, there must exist infinitely many such n.

Solution. Let S be the convex hull of the set of points (n, a_n) for $n \geq 0$. Geometrically, S is the intersection of all convex sets (or even all half-planes) containing the points (n, a_n); algebraically, S is the set of points (x, y) which can be written as $c_1(n_1, a_{n_1}) + \cdots + c_k(n_k, a_{n_k})$ for some $c_1, \dots, c_k$ which are nonnegative of sum 1.

We prove that for infinitely many n, (n, a_n) is a vertex on the upper boundary of S, and that these n satisfy the given condition. The condition that (n, a_n) is a vertex on the upper boundary of S is equivalent to the existence of a line passing through (n, a_n) with all other points of S below it. That is, there should exist a real number $m > 0$ such that

$$a_k < a_n + m(k - n) \qquad \forall k \geq 1. \tag{1}$$

We first show that $n = 1$ satisfies (1). The condition $a_k/k \to 0$ as $k \to \infty$ implies that $(a_k - a_1)/(k - 1) \to 0$ as well. Thus the set $\{(a_k - a_1)/(k - 1)\}$ has an upper bound m, and now $a_k \leq a_1 + m(k - 1)$, as desired.

Next, we show that given one n satisfying (1), there exists a larger one also satisfying (1). Again, the condition $a_k/k \to 0$ as $k \to \infty$ implies that $(a_k - a_n)/(k - n) \to 0$ as $k \to \infty$. Thus the sequence $\{(a_k - a_n)/(k - n)\}_{k>n}$ has a maximum element. Suppose $k = r$ is the largest value that achieves this maximum, and put $m = (a_r - a_n)/(r - n)$. Then the line through (r, a_r) of slope m lies strictly above (k, a_k) for $k > r$ and passes through or lies above (k, a_k) for $k < r$. Thus (1) holds for $n = r$ with m replaced by $m - \epsilon$ for suitably small $\epsilon > 0$.

By induction, (1) holds for infinitely many n. For any such n there exists $m > 0$ such that for $i = 1, \ldots, n - 1$, the points $(n - i, a_{n-i})$ and $(n + i, a_{n+i})$ lie below the line through (n, a_n) of slope m. That means $a_{n+i} < a_n + mi$ and $a_{n-i} < a_n - mi$; adding these together gives $a_{n-i} + a_{n+i} < 2a_n$, as desired.