Picture Perfect

5 portraits

A picture is worth a thousand words. We could not agree with this adage more, especially when it comes to illustrating mathematical ideas! From depicting the statement of the Pythagorean Theorem or the fact that the sum of the first n odd numbers is n^2, to illuminating proofs of algebraic statements in "proofs without words," well-drawn figures elucidate mathematics in helpful and beautiful ways. In this chapter, we share five *Playground* problems whose diagrams embody this spirit.

We begin with a problem from February 2001 posed by RGee Watkins. RGee was a maker of wooden puzzles from Hemet, California. His work was commemorated by one of the exchange puzzles from the International Puzzle Party (IPP) 28 held in Prague in 2008. A dozen of his puzzles are now on display at Indiana University as part of the Slocum Puzzle Collection, previously owned by Jerry Slocum, whom we meet in the Games chapter.

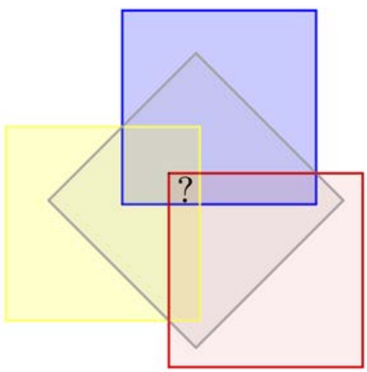

Figure 4. Overlapping squares

Problem S-48 (Vennlike Squares). Four overlapping squares intersect in ten regions as shown in Figure 4. Each of the ten regions is labeled with a distinct integer from 1 to 10. The sum of the labels inside each square is the same. What is the label of the unique region contained in all four squares (shown with a question mark in Figure 4)? ?

Two years later in September 2003, *The Playground* received the next problem from Avni Pllana of Feldkirch, Austria. The submission was unusual in two ways: first, it was the only communication Avni ever made to *The Playground*, and second, it was based almost entirely on a rather sophisticated diagram for the time, shown in Figure 5.

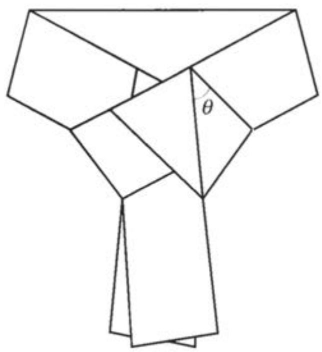

Figure 5. Paper folded tie

Naturally, this led to some curiosity on the part of the editors as to whether the tie can physically be constructed. If you're struggling with this yourself, there is a more modern diagram in Figure 6, shaded so that what we might call the front and back of the strip are burgundy and tan, respectively.

The modern diagram also reveals that the line segment just to the left of the label θ in Figure 5 is not a visible crease or paper edge, but rather an indication of a hidden paper edge below the top layer, to set the angle sought in the problem. (The presence of this segment in the original diagram did lead to some consternation in trying to reproduce the artifact.)

If that does not provide enough evidence, Figure 7 shows a photograph of this design actually folded from a strip of vellum, to show a hint of the hidden layers as well. It is gratifying that if the knots of this tie are all pulled tight yet kept flat, there are geometrically unique locations for all of the folds (except for the loop for the collar in back, which can be made lopsided rather than symmetric). In any case, on to the problem itself:

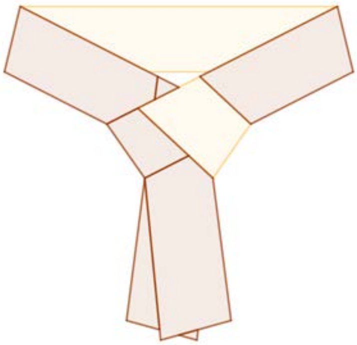

Figure 6. More detailed folding diagram for tie

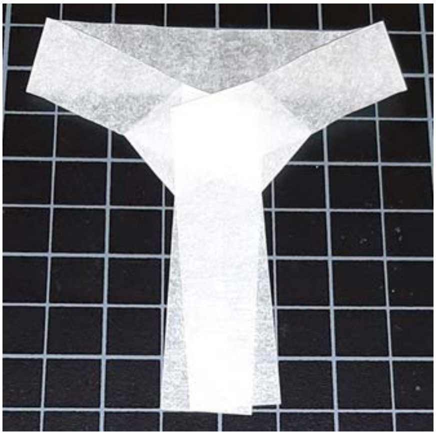

Figure 7. A vellum tie

Problem 176 (Paper Tie). A strip of paper is folded into a tie as shown in Figure 5. Prove that the angle θ is uniquely determined, and find its value, at least to a good numerical approximation, if not exactly. ?

Jerry Lo and David Rhee submitted a beautiful diagram with S-116, The Trilinear Pole, in February 2007. Jerry was a student at Ross Sheppard High School in Edmonton, where he was awarded the Pacific Institute for the Mathematical Sciences Fellowship for his performance on the 51st Alberta High School Mathematics Competition. David had been an undergraduate at the University of Waterloo and had been involved in research with Andy Liu for two years. In addition to this problem, which they also submitted a year earlier to *Integral*, Jerry and David composed two other problems together.

Before presenting their geometric challenge, we need the statement of Problem 11 on page 72 of Howard Eves' treatise *A Survey of Geometry* [**25**].

> Let ABC be a triangle. Let P be a point such that the lines AP, BP, and CP intersect the lines BC, CA, and AB at D, E, and F respectively. Let the lines EF, FD, and DE intersect BC, CA, and AB at L, M, and N, respectively, as shown in Figure 8. Prove that L, M, and N are collinear.

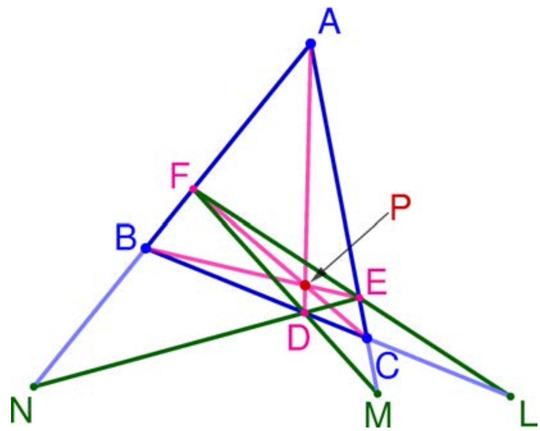

Figure 8. Trilinear pole

A line through L, M, and N would be called the *trilinear polar* of P with respect to triangle ABC, and P would be the *trilinear pole* of such a line.

Problem S-116 (The Trilinear Pole). Let p be a line that does not pass through any vertex of triangle ABC. Give a Euclidean construction of the trilinear pole of p with respect to ABC. ?

The mathematics department at Lafayette College in Easton, PA, produced numerous posers and solvers of *Playground* problems, no doubt because Derek Smith and Gary Gordon hailed from there. In April 2013, Rob Root, a member of this community, created a beautiful computer graphic illustrating a surprising trigonometric fact that lead to our next problem. When asked about its creation, Rob shared:

> It started with a simple insight: the sequence $a_n = \sum_{k=1}^{n} \sin(k)$ can be connected to the sequence $b_n = \sum_{k=0}^{n} \cos(k)$ by the recursion relations
>
> $$b_{n+1} = 1 + \cos(1)b_n - \sin(1)a_n \quad \text{and}$$
> $$a_{n+1} = 0 + \sin(1)b_n + \cos(1)a_n.$$

Gary and I pretty quickly worked out that this gives the lim sup and lim inf for the partial sums, and Thomas Yuster [another Lafayette math colleague whom we'll meet in the Opening Acts chapter] pointed out to us that putting this in the complex plane makes it even more elegant. Finally, Liz McMahon [yet a fourth Lafayette mathematician] asked about the interlocking sinusoids that appear in the plots of the sequence entries. These are the result of the rational approximations of π. My greatest benefit in working on this problem was having wonderfully creative and helpful colleagues!

It is also worth pointing out connections: I first stumbled on the recursion relations while thinking about Euler's product formula for the Riemann zeta function. I am not sure how similar the recursion relation is, but that is how I started hunting for it.

One of the first facts of trigonometry is that the sine function is bounded between -1 and 1. But, remarkably, the sequence $a_n = \sum_{k=1}^{n} \sin k$ is also bounded. The first 1,000 entries of this sequence produce the lovely graph, with surprising structure, shown in Figure 9.

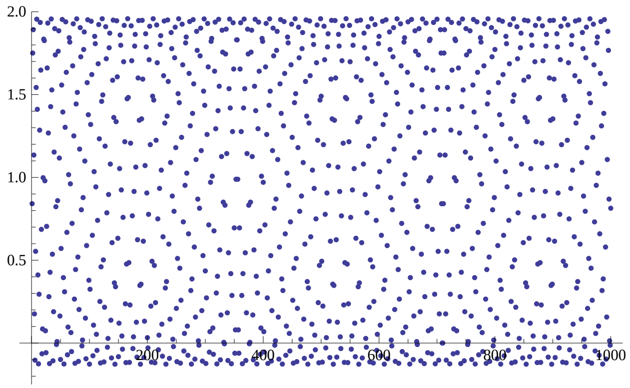

Figure 9. Sine sequence

Problem 293 (Warning Sines). Find the shortest closed interval $[r, s]$ such that $a_n \in [r, s]$ for all $n \geq 1$. ?

Reminiscing about the problem, Rob said:

All these years later, this problem offers plenty of threads to pull on. I think of it every time I see Vi Hart's tantalizing YouTube series on Fibonacci numbers and plants. Instead of using $e^i = e^{2\pi i/(2\pi)}$, she uses

$e^{2\pi i/\phi}$ to describe placements of leaves around a stem (where ϕ is the golden ratio).

This problem actually came to mind recently as I was reading Jordan Ellenberg's book *Shape* [**22**] for my first-year writing seminar at Lafayette. [We'll meet Jordan in the Prominent Players chapter.] The chapter on eigenvalues, "The Smoke in the Leaf," includes a lovely passage about the relationship between the golden ratio and the Fibonacci numbers that includes a conversational description of the proof that the multiples of ϕ (or any irrational number) mod 1 are dense on the unit interval [the Weyl-Bohl-Sierpiński Equidistribution Theorem]. Ellenberg attributes the proof to Dirichlet, but his description is so crystal clear, it deserves notice on its own. [It's on pp. 272–5 if you want to see it for yourself.] This is a slight variation on the idea that the integers mod 2π are dense on an interval of length 2π, and that of course is crucial in the proof that the lim sup and lim inf are approached arbitrarily closely by partial sums in this problem.

We wrap up this chapter with the work of Arsalan Wares, the most prolific creator of beautiful geometric problems for *The Playground* to date. Arsalan served as a Professor of Mathematics Education at Valdosta State in Georgia and has YouTube channel dating back to 2007 with over 4,000 viewers.

To give you an idea of Arsalan's range, see the gallery of some of his submitted *Playground* images below. In many cases, you can see what the problem is simply from the illustration!

Problems by Arsalan Wares

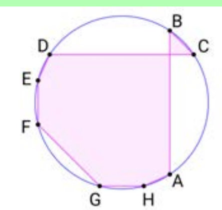

Figure 1. An example cyclocts. A point is in its interior if you must cross its edges an odd number of times to reach the circle.

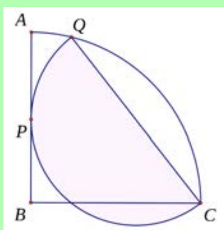

Figure 1. A semicircle almost inscribed in a quarter circle.

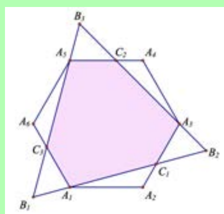
Figure 1. An overlapping hexagon and triangle of equal area.

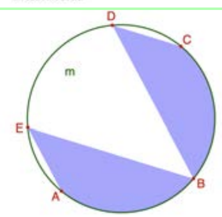

Figure 1. Circle-M ranch logo.

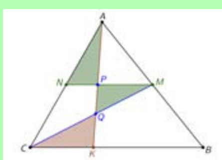

Figure 1. Three smaller triangles formed inside a larger one.

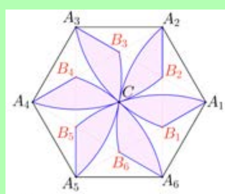

Figure 1. A flower inscribed in a hexagon.

We're delighted to conclude with a brand-new problem supplied by Arsalan specifically for this volume. When asked about its creation, he recounted:

> I do origami; to do origami, I need origami paper. Instead of buying origami paper, I try to design my own. I have a penchant for geometric motifs. I was trying to draw teardrop shapes inside a circle. Many of us are familiar with the shape that involves two teardrops inside a circle, a version of which appears in the South Korean flag. I have also seen motifs with five teardrops inside a circle in Japanese art. Once I figured out how to draw five teardrops inside a circle accurately, I wondered about drawing eight or nine teardrops inside a circle. Eight teardrops can be drawn inside a circle in a similar manner. The next question that came to my mind was more mathematical in nature: What fraction of the circle is covered by the eight teardrops? Geometric problems with artistic shapes appeal to me; these problems blur the line between mathematics and art. But again, is there really a line between mathematics and art?

Problem X-5 (Eight Teardrops). Eight congruent circular arcs are touching the inside of a circle, and each arc also touches the two adjacent arcs, as shown in Figure 10. All arcs and circles are tangent wherever they touch. Each region bounded by two neighboring arcs and the outer circle has been shaded pink. What fraction of the area of the circle has been shaded pink? ?

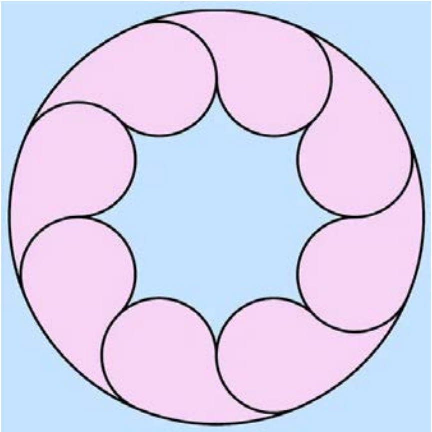

Figure 10. Eight identical arcs nestled in a circle