Preface

This volume was prepared after the successful completion of the 2020 AMS Short Course called "Mean Field Games: Agent Based Models to Nash Equilibria", given in Denver on January 13 and 14, 2020. All the chapters were written by the speakers.

Mean field game theory was initiated more than fifteen years ago by two groups of researchers: Jean-Michel Lasry and Pierre-Louis Lions on the one hand, and Peter Caines, Minyi Huang and Roland Malhamé on the other hand. The subject, at the intersection between mathematics and complex systems, has been extraordinarily successful and has continued to grow since then. It now features various developments in different areas of mathematics (probability theory, stochastic control, partial differential equations, calculus of variations, optimal transport...) and calls for innovative numerical solutions, with connections with other scientific fields. The six chapters that make up this volume give an overview of the subject, from the foundations of the theory to the applications to economics and finance, including computer science aspects.

The primary objective of mean field game theory is to provide a robust methodology for studying large systems of interacting rational agents. Unlike many models of complex systems in which the dynamics of the agents are prescribed in advance, mean field games offer a more general framework in which the agents have the ability to make decisions. Decisions are made by each agent to best satisfy an individual cost or reward based on the state of the others. The agents thus become players and the dynamics of the whole system can only be described a posteriori in the form of an equilibrium or a compromise between all the players. Mathematically, the model is thus a game and the search for solutions is notoriously computationally heavy as the number of players increases. The thrust of mean field game theory is to be scalable. The modeling is based on the mean field paradigm, inspired by statistical physics: the players show a form of statistical homogeneity and only see each other through aggregated quantities. The law of large numbers allows to just describe equilibria statistically: a mean field game aims at identifying the statistical state of the population with the law of the optimal response that a rational player in this population would choose. The solutions are said to be distributed: each player can implement the equilibrium strategy knowing only their own state.

The volume reviews several of the most important elements of the theory. It is organized into three sections of two chapters each. The first part covers the basics of the topic. Chapter 1, written by Roland P. Malhamé and Christy Graves, is a very good entry point. From practically motivated examples underpinning Roland Malhamé's earlier works, the reader will discover the basic principles of mean field game theory. The fixed point problem describing an equilibrium leads to a coupled system of two forward and backward in time equations: the forward equation describes the evolution of the population and the backward equation describes the evolution of the cost (or the reward) to a tagged player. Mathematically, solving this forward-backward system is the true spice of mean field games. The examples treated in Chapter 1 lead to finite-dimensional ordinary differential equations. The general version, which fits many of the examples from the literature, is presented in Chapter 2, which I wrote. In a generic way, the forward and backward equations become infinite-dimensional: In analysis, these two equations are written as partial differential equations. In probability theory, they are written as stochastic differential equations, called McKean-Vlasov equations, with the coefficients depending on the law of the solution. The reader will find in Chapter 2 the main lines of the study of these systems. The reader will also see that the forward and backward equations can be interpreted as the characteristic system of a partial differential equation set on the space of probability measures. The latter is called *master equation*. In this respect, Chapter 2 gives an overview of the differential calculus on spaces of probability measures. In this chapter, the link is also made with so-called mean field control problems, in which the players no longer compete but cooperate. Chapters 3 and 4 in Part II are devoted to the justification of the mean field formulation: the connection is rigorously established between games with a large number of players and mean field games, which is a very important question. In Chapter 3, the link is proven in several ways: It is shown how the solution of a mean field game could be inserted into a game with a finite number of players. It is also shown that, under appropriate assumptions, solutions of finite games converge to the solution of the corresponding mean field game when the number of players tends to infinity. This latter problem is known as the convergence problem and is reputed to be hard. The two main ways to solve it, one based on the master equation and the other one on compactness arguments, are explained in Chapter 3. Daniel Lacker, author of this chapter, is a recognized expert on these methods. A refined analysis of the fluctuations and the deviations in the convergence problem is provided in the next chapter. In particular, the novice reader will find in Chapter 4 prepared by Kavita Ramanan a very useful and pedagogical introduction to the theory of fluctuations and deviations for McKean-Vlasov equations, which is a prerequisite for the analysis in the broader context of mean field games. The last two chapters, in Part III, are of a more practical inspiration and complete in this sense the panorama offered to the reader. In particular, René Carmona gives, in Chapter 5, a peerless picture of the applications of mean field games to economics and finance. The reader will find numerous examples from the literature illustrating the scope of the theory. On this occasion, the reader will discover that several economics papers, prior to the mathematical works of Lasry and Lions and of Caines, Huang and Malhamé, actually already contained very similar ideas. The volume ends with Chapter 6, in which Mathieu Laurière exposes very clearly existing numerical methods for mean field games and mean field control problems, with an opening towards machine learning techniques, which the reader will obviously appreciate.

I could not conclude this preface without thanking all the authors who contributed to this volume: Christy, Kavita, Daniel, Mathieu, René and Roland, many thanks! I would also like to warmly thank the AMS editorial team as well as the team in charge of the AMS Short Course in Denver, especially Thomas Barr, Lori

Melucci and Christine Thivierge. My trip to Denver for the AMS Short Course will remain in my memory in a unique way: this was my last trip across the Atlantic before the first lockdown measures were implemented two months later. Due to the special conditions we faced then, the preparation of the volume was unexpectedly delayed, but our determination to complete the project has remained unfailing. We hope the reader will appreciate the result.

François Delarue
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Nice, France