Preface

This volume collects together the Proceedings of a (on-line) Conference at the Fields Institute (September 27-October 1 2021) dedicated to celebrating the 40-th anniversary of the birth announcement of cyclic cohomology at the Oberwolfach meeting in September 1981, where the talk "Spectral sequence and homology of currents for operator algebras" described the theory as well as its main properties. Cyclic cohomology was originally conceived as a tool in noncommutative differential geometry – an area of research, nascent at the time, dedicated to the exploration of geometric spaces whose algebras of coordinates are convolution algebras reflecting not only the points of the spaces but also their interrelations. These spaces arise naturally in pure mathematics and in quantum physics. Aside from the obvious examples of groupoids in differential geometry and phase space in quantum physics, there are some key examples of spaces where this new point of view applies; these are the micro structure of space-time and the Arithmetic Site underlying the geometry of the prime numbers.

Cyclic cohomology plays a fundamental role in an extensive machinery that allows for the formulation and investigation of the geometric properties of noncommutative structures. Given a noncommutative algebra A, one classifies the differential graded algebras (DGA) with A in degree 0 and which are endowed with a closed graded trace. These structures are characterized by the multilinear form τ on A giving the trace of the element $a_0 da_1 da_2 \dots da_n$ of the DGA. This multilinear functional, called the "character", fulfills two key properties: it is a Hochschild cocycle and it is cyclic. The cyclic cochains form a subcomplex of the Hochschild complex, whose cohomology is the cyclic cohomology of A. The key result is that K-homology classes on A define such a DGA with closed graded trace, and that there is a general index formula computing the Fredholm index in terms of the pairing of the cyclic cohomology character of the K-homology class with the straightforward K-theory Chern character.

The forgetful map from cyclic cohomology to Hochschild cohomology, together with the boundary operator B playing the role of the de Rham boundary of currents, are then part of a long exact sequence involving the periodicity operator S. The latter is defined by means of the tensor product with the cyclic cohomology of the scalar field: a polynomial ring in one generator of degree 2. These general facts provide the theoretical framework of numerous index formulas and they play a central role in noncommutative geometry.

The circle of ideas involving cyclic homology turned out to be important well beyond the core field of noncommutative geometry. In particular, it has a strong footing in number theory, algebraic geometry, and in homotopy theory (algebraic topology). Developments in all these areas are interconnected and influence each other. Indeed, the implementation of the purely algebraic definitions of cyclic (co)homology and their functional analytic variants directly inspired the construction of the highly successful topological Hochschild homology (THH) and topological cyclic homology (TC) of Bokstedt, Hsiang and Madsen in the world of ring spectra. In turn these advancements supplied a remarkable computational tool in algebraic K-theory. We refer to the survey of B. Dundas in this volume for a historical overview of this part. With respect to the previously existing Hochschild homology, the novelty implemented by the cyclic aspect is the natural circle action on the topological Hochschild homology. This provides a computational tool of great relevance through the implementation of trace maps providing relations between topological cyclic homology and algebraic K-theory.

From the start, and in parallel to algebraic K-theory, the development of cyclic theory has been highly interdisciplinary and transverse to the traditional classification of mathematical fields. This book of Proceedings gives an overview of the richness of the subject.

In algebraic topology, the cyclotomic structure obtained using the cyclic subgroups of the circle action on topological Hochschild homology gives rise to remarkably significant arithmetic structures intimately related to crystalline cohomology, through the de Rham-Witt complex, and to Fontaine's theory of periods. At the non-archimedean places, the work of L. Hesselholt has shown that using the cyclic structure on topological Hochschild homology (THH) and the additional properties due to replacing the integers by the sphere spectrum as a base, one recovers not only the Witt construction and the de Rham-Witt complex but also elements of Fontaine's theory. In his contribution to this volume, Hesselholt highlights some of the spectacular recent results on topological cyclic homology of Nikolaus-Scholze, Bhatt–Morrow–Scholze, and Antieau-Mathew-Morrow-Nikolaus and explains how the Fargues-Fontaine curve with its decomposition into a punctured curve and the formal neighborhood of the puncture, naturally appears from various forms of topological cyclic homology and maps between them. We refer to the contribution of B. Tsygan in this volume for the construction of the de Rham-Witt complex in the noncommutative framework.

Recently, there has been an extensive body of work done on Hochschild and cyclic homology in the framework of infinity categories. In his contribution to this volume, D. Gepner gives a user friendly introduction to higher algebra generalizing ordinary algebra, that is algebra in the setting of ordinary, discrete, set based category theory. On the other hand, the contribution of R. Meyer and D. Mukherjee is part of a program to define analytic cyclic homology theories for bornological algebras over non-archimedean fields.

This book also reports a number of exciting developments in the complex theory. An active area of research exhibits the role of cyclic homology in higher index theory and the topology of manifolds. We refer to the survey article of P. Piazza and X. Tang on their work on the pairings of cyclic cocycles with secondary K-theory index class and the applications to Atiyah-Patodi Singer index theory in the context of proper Lie group actions. As a by-product, they obtain an interesting analog of the Connes-Moscovici higher index theorem for manifolds with boundaries in the context of proper Lie group actions. The analytic difficulty of capturing the invariance of higher signatures (or the vanishing of indices due to positive scalar curvature) in terms of cyclic cohomology index formulas consists in passing from the framework of topological K-theory of C^*-algebras, where the invariance takes place, to the cyclic cohomology of appropriate subalgebras suitable for differential geometric purposes. In their contributed article J. Wang, Z. Xie, and G. Yu introduce the new ℓ^1-index theory and its pairing with cyclic cohomology for both closed manifolds and compact manifolds with boundary, and prove an ℓ^1-version of the higher Atiyah-Patodi-Singer index theorem for manifolds with boundary. This article opens a new domain of exploration.

Another very active research domain highlighted in this book is the computation of index cocycles of pseudodifferential operators in terms of noncommutative residues and the replacement of abstract K-theory statements in index theory by concrete formulas with an unlimited range of applications. This is illustrated in several contributions, as follows. A. Savin and E. Schrohe extend the residue and trace expansions to the algebra of Fourier integral operators on Euclidean space. A. Gorokhovsky and E. van Erp obtain a formula for the index of a pseudodifferential operator with invertible principal symbol in the extended Heisenberg calculus of Epstein and Melrose. Furthermore, the contribution of J. Block, N. Higson and J. Sanchez presents an overview of the work of Perrot on new index cocycles.

Several articles address the problem of computing the cyclic homology of algebras occurring naturally in noncommutative geometry. R. Ponge presents a survey of the large amount of results on the cyclic homology of cross products by group actions, and M. Puschnigg gives a new method to obtain the periodic cyclic homology of crossed products for actions of discrete groups directly, without passing through Hochschild and cyclic homology. In his contribution, M. Lorentz adresses the Hochschild cohomology of the uniform Roe algebras. Y. Song and X. Tiang compute the variation of orbital integrals for the Cartan motion group under the deformation groupoid which embodies the Mackey principle in representation theory of Lie groups. The work of M. Pfaum applies methods of complex algebraic geometry to obtain general localization results in noncommutative geometry, by sheafification of the algebras under consideration and reduction of the computation to the stalks of the sheaf. Finally A. Baldare M. Benameur and V. Nistor provide a general principle which allows one to compare cyclic homology with K-theory after tensoring by scalars.

In the framework of noncommutative geometry the group symmetries are extended to Hopf algebra actions, while Lie algebra cohomology is greatly extended to the cyclic cohomology of Hopf algebras, which becomes the natural receptacle for characteristic classes. For this fundamental aspect of the general theory we refer to the contribution of H. Moscovici on the van Est analogy in Hopf cyclic cohomology, and to the extensive survey of M. Khalkhali and I. Shapiro. Both these articles open new directions of research.

The development of index theory in the noncommutative context has shown that the most embracing notion of geometric space suitable to develop the general theory is that of a smooth groupoid. We refer to the article of P. Carillo Rouse for the construction of the assembly map in this context.

The quantized calculus associated with a K-homology class was the key construction in the original development of the theory: we refer to the contribution of E. McDonald, F. Sukochev and X. Xiong for their work on the delicate analytic properties involved. The link between the characters of the obtained DGAs generated the periodicity operator S in cyclic cohomology. We refer to the contribution of J. Cuntz for the comparison of S with Bott periodicity.

The impact of cyclic theory on physics is reported in two fundamental contributions. In solid state physics, it is described by the contribution of E. Prodan on cyclic cocycles and quantized pairings in materials science. In quantum field theory, it is explained in the breakthrough contribution of T. van Nuland and W. van Suijlekom, where the perturbations of the spectral action are amazingly encoded by an entire cocycle whose cyclic properties survive at the one loop level and are intimately related to the Ward identities of perturbative expansions of gauge theories.

Finally, as a far-reaching goal, one can envisage that cyclic homology (in some guise of deRham-Witt or crystalline cohomology) should give access to the sought-for cohomology of the Arakelov compactification of the spectrum of the integers. On this topic, the joint contributions of A. Connes and C. Consani describe the way the zeros of the Riemann zeta function appear using the trace map on the noncommutative adèle class space, and the role of cyclic homology in the arithmetic context, after introducing a suitable structure sheaf for the Arakelov compactification of the spectrum of the integers. The enrichment of the cyclic theory due to working over the absolute base (i.e. the sphere spectrum, here understood in its primitive categorical characterization), refines the combinatorial backbone provided by the cyclic category, through the introduction of the new pericyclic category, explicitly involving the cyclotomic structure.