Introduction

Ideas from quantum field theory and string theory have had an enormous impact on geometry over the last two decades. One extremely fruitful source of new mathematical ideas can be traced back to works of Cecotti, Vafa, and their coauthors around 1991 on the Geometry of Topological Field Theory. The motivation for their "tt*-geometry" came from physics, but the work turned out to unify ideas that had been developing in such separate branches of mathematics as singularity theory, Hodge theory, integrable systems, matrix models, and Hurwitz spaces.

From one point of view, the geometry developed by Cecotti, Vafa, et al involved a refinement of the notion of Higgs bundle. This geometric structure had been previously introduced and studied by Hitchin. The refinement was closely related to what have since been codified by Dubrovin as Frobenius manifolds. Essentially the same objects had been previously introduced, on moduli spaces asscociated to quasihomogeneous singularities, by Kyoji Saito. They appeared also in Hodge theoretic work of Morihiko Saito and in numerous other approaches. In the papers by Cecotti, Vafa et al, these purely holomorphic structures were combined with a generalization of the Gauss -Manin connection, a generalization which nowadays is described as variations of harmonic Higgs bundles. Following ideas of Hitchin, Deligne and Simpson, the entire structure is most conveniently described by introducing a twistor line and a new, flat connection on the resulting space. The flatness equations for this connection are the tt*-equations of Cecotti and Vafa. The twistor structures capture the essence of the Corlette-Simpson non-Abelian Hodge theory, which establishes the equivalence of flat connections and Higgs bundles. The tt*-geometry combines algebro-geometric notions also with symplectic ones. Such a combination turns out to be tailor made for investigations of mirror symmetry and Gromov-Witten invariants.

These structures and equations have since appeared in many other branches of geometry and physics: Integrable systems have been closely related to the topic ever since Hitchin's works on moduli spaces of stable bundles, and Dubrovin has proved integrability of the tt*-equations. Almost simultaneously with the original work by Cecotti and Vafa, Simpson developed his notion of harmonic bundles. These have since evolved into mixed twistor structures, an essential contritubion to the field. Another key ingredient is Mochizuki's ground breaking work on tame –and more recently also wild– harmonic bundles. Hertling and collaborators have clarified and further developed the precise mathematical counterparts of the fundamental structures used in the physics papers of Cecotti and Vafa, introducing "TERP structures". On the physics side, the tt*-equations have been understood as governing tree level amplitudes in topological quantum field theories. Their generalization to higher-genus amplitudes by Bershadsky-Cecotti-Ooguri-Vafa is known as the holomorphic anomaly equation. The latter provides a recursive tool to calculate some of the quantities that are of interest in topological field theories – like the Gromov-Witten invariants and some of their cousins. The impact on enumerative geometry has been tremendous, and the full scope of ideas is yet to be understood.

The present volume is loosely based on the workshop "From tQFT to tt* and integrability", which we organized at the University of Augsburg in May 2007. We have tried to combine some of the above-mentioned, so far only partially understood aspects into a cohesive picture, and this proceedings volume aims to give an overview of these topics, leading to and including up-to-date research.

The first four papers of this volume give an overview of the underlying geometric structures along with recent developments of the latter. Claude Sabbah explains the notion of tt*-geometry in terms of a common extension of Saito structures and harmonic Higgs bundles. The existence of a canonical harmonic structure on the canonical Frobenius manifold attached to a convenient and nondegenerate Laurent polynomial is explained – a result that appears implicity already in the work of Cecotti and Vafa. The following paper by Kyoji Saito and Atsushi Takahashi addresses the holomorphic part of tt*-geometry in the context of singularity theory. The article gives a historical review and introduction to the ideas which led to the discovery of the structure of a Frobenius manifold on the base of the universal unfoldings of isolated singularities by means of primitive forms. Claus Hertling and Christian Sevenheck explain the full tt*-geometry for isolated hypersurface singularities and tame functions. They give the technical construction of all relevant geometric ingredients, introducing TERP structures, and explaining how the latter give rise to variations of twistor structures, which in turn they relate to harmonic bundles. Moreover, nilpotent orbits are discussed, and appropriate classifying spaces and limiting structures are introduced. The contribution by Vicente Cortés and Lars Schäfer emphasizes differential geometric aspects of tt*-geometry. A classical result of Dubrovin's, which describes solutions to the tt*-equations in terms of pluriharmonic maps, is explained and generalized. In particular, the relation of tt*-geometry to various other special geometries is worked out in detail.

All these diverse notions come together in the paper by Ludmil Katzarkov, Maxim Kontsevich and Tony Pantev. They generalize Hodge structures and their variation by developing an abstract theory of non-commutative Hodge structures, investigating existence and variations, and proposing explicit construction and classification techniques. Their work generalizes the algebraic constructions of Saito and Takahashi (as explained in the appendix to [Saito-Takahashi]) as well as the results from Hodge and singularity theories in the [Hertling-Sevenheck] and [Cortés-Schäfer] works. This combines tt*-geometry with the notions of weight systems, harmonic and wild bundles, and pure twistor structures, semi-infinite Hodge structures, and TERP structures. Moreover, connections to mirror symmetry and its applications are explicitly explained.

The Hodge theoretic approach is central also to the next contribution. Carlos Simpson studies the simplest cases (rank 1) of twistor spaces on open varieties. He points out that the space of local monodromy transformations around a divisor component at infinty has weight two – a new phenomenon in the non-compact case, suggesting one of the essential difficulties involved in constructing a complete mixed twistor theory.

The remaining papers of the volume focus on applications or generalizations of the ideas originally described by Cecotti, Vafa, and collaborators to integrability and to enumerative geometry.

The relation to integrable systems is the topic of the contributions by Luuk Hoevenaars and by Anton Gerasimov and Samson Shatashvili. While the former explains how the base of a special integrable system - the Neumann system - carries the structure of an almost Frobenius manifold, the latter gives evidence for the proposal that a certain quantum integrable system captures the geometry of the moduli space of Higgs bundles in an appropriate setting.

The paper by Vincent Bouchard and Marcos Mariño makes use of recent progress in the matrix model approach to topological field theories due to Eynard and Orantin. From it, the authors derive a conjectured recursive solution for Hurwitz numbers at all genera. Andrew Neitzke and Johannes Walcher study a further generalization of the tt*-equations beyond the Bershadsky-Cecotti-Ooguri-Vafa anomaly equations, namely to a setting on Calabi-Yau threefolds with background D-branes. The corresponding open topological partition functions are obtained by a shift of variables from the solutions to the ordinary holomorphic anomaly equations. This yields a proposal which interprets an auxiliary Hilbert space introduced by Witten as the space of states generated by all open topological partition functions.

Acknowledgements. We are grateful to the authors for the great effort they made to balance presentation of their latest results with careful introductions to the many facets of this deeply interconnected and fast evolving subject. We are also grateful to all the speakers and participants of the Augsburg conference "From tQFT to tt* and integrability", May 25-29, 2007. The conference was funded by NSF Focused Research Grant DMS 0139799 - The Geometry of Superstrings - of the Penn Math/Physics group (RD, Principal Investigator), and by DFG grant GZ 4851/142/07 of KW at the University of Augsburg.

We are grateful to the editors at the AMS for their help in putting this volume together: We cordially thank Sergei Gelfand and Christine Thivierge for their continuous guidance and help. Arthur Greenspoon has helped us with the editorial fine-tuning of every contribution and deserves our special gratitude.

Ron Donagi and Katrin Wendland