On the scientific work of V.G. Maz'ya: a personalized account

Vladimir Gilelevich[1] Maz'ya, one of the most distinguished analysts of our time, has recently celebrated his 70th birthday. This personal landmark is also a great opportunity to reflect upon the depth and scope of his vast, multi-faceted scientific work, as well as on its impact on contemporary mathematics.

It is no easy task to re-introduce to the general public a persona of the caliber of Vladimir Maz'ya. Nonetheless, the narrative of his life is such an inspirational epic of triumph against adversity and seemingly insurmountable odds, of sheer perseverance and dazzling success, that such an endeavor is worth undertaking even while fully aware that the present abridged account will have severe inherent limitations.

Siméon Poisson once famously said that "*life is good for only two things: discovering mathematics and teaching mathematics*". Considering the sheer volume of his scientific work and scholarly activities, one might be tempted to regard Vladimir Maz'ya as the perfect embodiment of this credo. However, with his larger-than-life personality, boundless energy, strong opinions and keen interest in a diverse range of activities, Vladimir Maz'ya transcends such a cliche: he is a remarkable man by any reasonable measure. His life, however, cannot be separated from mathematics, regarded as a general human endeavor: much as his own destiny has been prefigured by his deep affection for mathematics, so has Vladimir Maz'ya helped shape the mathematics of our time. Meanwhile, his views on mathematical ability are rooted in a brand of stoic pragmatism: he regards the latter not unlike the skill and sensitivity expected of a musician, or the stamina and endurance required of an athlete. In [**6**], I. Gohberg remarks: "*Whatever he writes is beautiful, his love for art, music and literature seeming to feed his mathematical aesthetic feeling*".

I. Rough childhood. Vladimir Maz'ya was born on December 31, 1937, in Leningrad (present day St. Petersburg) in the former USSR, roughly two years before World War II broke out in Europe. USSR was subsequently attacked and the capture of Leningrad was one of three strategic goals in Hitler's initial plans for Operation Barbarossa ("Leningrad first, the Donetsk Basin second, Moscow third"), with the goal of "Celebrating New Year's Eve 1942 in the Tsar's Palaces." It is in this context that Vladimir Maz'ya's early life was marred by profound personal tragedy: his father was killed on the World War II front in December 1941, and all four of his grandparents perished during the subsequent siege of Leningrad, which lasted from September 9, 1941 to January 27, 1944. Vladimir was brought up by his mother, alone, who worked as a state accountant. They lived on her

[1] patronymic after his father Hillel

meager salary in a cramped (nine square meter) room of a big communal apartment. These days, it is perhaps difficult to imagine the hardship in which a young Vladimir was finding his feet, and yet he spoke of occasional glimmers in this desolate atmosphere. He once recounted a touching story about the lasting impression a children's botanical book he had received, about the fruits of the world, made on him: how the pictures he gazed upon over and over still vividly live in his memory, and how it took many long years before he had a chance to actually see and taste some of the fruits depicted there. Resolute and driven, Vladimir rose above these challenges. At the same time, his talent and ability were apparent from early on: he earned a gold medal in secondary school and, as a high-schooler, he was a frequent winner of city olympiads in mathematics and physics.

II. The formative years. While 17 years of age, Vladimir Maz'ya entered the Faculty of Mathematics and Mechanics (Mathmech) of Leningrad State University (LSU) as a student. His first publication, "On the criterion of de la Vallée-Poussin", was in ordinary differential equations and appeared in a rota-printed collection of student papers when he was in his third year of undergraduate studies. In the following year, while he was a fourth-year student, his article on the Dirichlet problem for second order elliptic equations was published in *Doklady Akad. Nauk SSSR.* Upon finishing his undergraduate studies at Mathmech-LSU, Vladimir Maz'ya secured a position as a junior research fellow at the Research Institute of Mathematics and Mechanics of Leningrad State University. Two years later he successfully defended his Ph.D. thesis on "Classes of sets and embedding theorems for function spaces". This remarkable piece of work was based on ideas emerging from his talks in Smirnov's seminar. In their reviews, the examiners noted that the level of quality and technical mastery far exceeded the standard requirements of the Higher Certification Commission for Ph.D. theses. Testament to the outstanding nature of his thesis, Vladimir Maz'ya was awarded the Leningrad Mathematical Society's prize for young scientists. Subsequently, Vladimir Maz'ya was a volunteer director of the Mathematical School for High School Students at Mathmech, an institution born out of his own initiative. Interestingly, Vladimir Maz'ya never had a formal scientific adviser, both for his diploma paper (master's thesis), and for his Ph.D. thesis. Indeed, in each instance, he chose the problems considered in his work by himself. However, starting with his undergraduate years, he became acquainted with S.G. Mikhlin, and their relationship turned into a long-lasting friendship that had a great influence on the mathematical development of Vladimir Maz'ya. According to I. Gohberg, [**6**], "*Maz'ya never was a formal student of Mikhlin, but Mikhlin was for him more than a teacher. Maz'ya had found the topics of his dissertations by himself, while Mikhlin taught him mathematical ethics and rules of writing, refereeing and reviewing.*"

III. Becoming established. During 1961-1986, Vladimir Maz'ya held a senior research fellow position at the Research Institute of Mathematics and Mechanics of LSU. Four years into that tenure, he defended his D.Sc. thesis, entitled "Dirichlet and Neumann problems in domains with non-regular boundaries", at Leningrad State University. From 1968 to 1978, he lectured at the Leningrad Shipbuilding Institute, where he became a professor in 1976. In 1986 he departed the university for the Leningrad Division of the Institute of Engineering Studies of the Academy of Sciences of the USSR, where he created and headed the Laboratory of Mathematical Models in Mechanics. At the same time, he also founded the influential Consultation Center in Mathematics for Engineers, serving as its head for several years. In 1990 Vladimir Maz'ya relocated to Sweden and became a professor at Linköping University. At this stage in his career, in recognition of his fundamental contributions to the field of mathematics, Vladimir Maz'ya has become the recipient of a series of distinguished awards in relatively quick succession. In 1990 he received an honorary doctorate from the University of Rostock, Germany. In 1999 he was the recipient of the Humboldt Prize, and in 2000 was elected a corresponding member of the Royal Society of Edinburgh (Scotland's National Academy). Two years later he became a full member of the Royal Swedish Academy of Sciences. In 2003 he received the Verdaguer Prize of the French Academy of Sciences, and in 2004 the Celsius Gold Medal of the Royal Society of Sciences at Uppsala. A number of international conferences in his honor have been organized during this period of time, such as the conference in Kyoto, Japan, in 1993, the conferences at the University of Rostock, Germany, and at École Polytechnique, France, in 1998, and the conferences in Rome, Italy, and Stockholm, Sweden, in the summer of 2008. In 2002 Vladimir Maz'ya was an invited speaker at the International Congress of Mathematicians in Beijing, China. More recently, he has held appointments at the University of Liverpool, England, and at the Ohio State University, USA, while continuing to be a Professor Emeritus at Linköping University, Sweden.

IV. The mathematical work. By any standards, Vladimir Maz'ya has been extraordinarily prolific, as his 50 years of research activities have culminated in about a couple dozen research monographs, and more than 450 articles, containing fundamental results and powerfully novel techniques. Besides being remarkably deep and innovative, his work is also incredibly diverse. Drawing upon several sources, most notably [**1**], [**2**], [**5**] and [**9**], below we briefly survey some of the main topics covered by Vladimir Maz'ya's publications. The references labeled [Ma-X] refer to the list of books published by Vladimir Maz'ya, which is included following the current subsection.

◇ `Boundary integral equations on non-smooth surfaces.` One of the early significant contributions of Vladimir Maz'ya was his 1967 monograph [Ma-25] with Yuri D. Burago, where they developed a theory of boundary integral equations (involving operators such as the harmonic single- and double-layer potentials) in the space C^0, of continuous functions, on irregular surfaces. The book contains two parts: the first of which concerns the higher-dimensional potential theory and the solutions of the boundary problems for regions with irregular boundaries, while the second part deals with spaces of functions whose derivatives are measures. This was happening around the time the Calderón-Zygmund program, one of its goals being a re-thinking of the finer aspects of Partial Differential Equations from the perspective of Harmonic Analysis, was becoming of age. In the early 60's, the solvability properties of elliptic multidimensional singular integral operators were well-understood, due to the fundamental contributions of people such as Tricomi, Mikhlin, Giraud, Calderón and Zygmund, and Gohberg, among others; but very little was known about the degenerate and/or non-elliptic case. Influenced by Mikhlin, Vladimir Maz'ya began in the mid 60's a life-long research program (part of which has been a collaboration effort) aimed at shedding light on this challenging and important problem. These innovative ideas did not get instantaneous recognition as a certain degree of skepticism has long accompanied efforts to understand non-smooth calculus. One well-known quotation attributed to H. Poincaré, which typifies the aforementioned distrust, goes as follows: "*Autrefois quand on invantait une fonction nouvelle, c'était en vue de quelque but practique; aujourd'hui on les invente tout exprès pour mettre en défaut les raisonnements de nos péres et on n'en tirera jamais que celà*". Such a point of view was by no means isolated. Even S.G. Mikhlin, years later, referring to the perspective of studying PDE's under minimal smoothness assumptions on the boundary, opined to the effect that "*no mother would ever let her child play in such ravines*".

The subject of analysis in non-smooth settings permeates through much of the work of Vladimir Maz'ya, who has had a most significant contribution in ensuring the eventual acceptance of this, nowadays fashionable, area of research. In collaboration with his Ph.D. student N.V. Grachev (1991), Vladimir Maz'ya solved the classical problem of inverting the boundary integral operators naturally associated with the Dirichlet problem for the Laplacian, in the space C^0, on a polyhedral surface. Also, Maz'ya and A. A. Solov'ev were the first to consider (in 1990) boundary integral equations on a curve with cusps. Subsequently, they developed a logarithmic potential theory which is applicable to integral equations in elasticity theory in a plane domain with inward or outward peaks on the boundary (2001). More recently, in collaboration with T. Shaposhnikova, Vladimir Maz'ya has studied the classical boundary integral equations of the harmonic potential theory on Lipschitz surfaces, and obtained higher fractional Sobolev regularity results for their solutions under optimal regularity conditions on the boundary. The method employed, going back to work of Maz'ya in the early 80's, consists of establishing well-posedness results for certain auxiliary boundary value and transmission problems for the Laplace equation in weighted Sobolev spaces.

◇ `Counterexamples related to Hilbert's 19th and 20th problems.` In his famous plenary address at the International Congress of Mathematicians in 1900, held at the Sorbonne, Paris, David Hilbert put forth a list of twenty-three open problems in mathematics, many of which turned out to be very influential for 20th century mathematics (strictly speaking, Hilbert presented ten of the problems: 1, 2, 6, 7, 8, 13, 16, 19, 21 and 22, at the conference, and the full list was published later). The 19th problem read: *Are the solutions of regular problems in the calculus of variations always analytic*? Originally, Hilbert was referring to regular variational problems of first order in two-dimensional domains, but the issue of (local) regularity makes sense in higher dimensions and for higher-order problems as well. Hilbert's 19th and 20th problems, the latter asking "*Is it not the case that every regular variational problem has a solution, provided certain assumptions on the boundary conditions are satisfied, and provided also, if need be, that the concept of solution is suitably extended?*" have generated a large amount of attention and, in the second half of the 20th century, proofs were obtained in sufficient generality. It was therefore natural to speculate that the conjectures continue to hold for *higher-order* variational problems. However, in 1968 Vladimir Maz'ya proved that this is not the case. In [**8**], Maz'ya constructed higher-order quasi-linear elliptic equations with analytic coefficients whose solutions are not smooth.

Other counterexamples constructed in [**8**] (and, independently, by De Georgi [**4**]) concern the celebrated De Giorgi-Nash Hölder regularity result for solutions of the second order linear elliptic equations in divergence form with bounded measurable coefficients. Maz'ya showed that this property fails for higher-order equations which may admit variational solutions which are not locally bounded. The counterexamples in [**8**] stimulated the development of the theory of partial regularity of solutions to nonlinear equations, i.e., the study of regularity properties outside of a sufficiently small, exceptional set.

◇ `The oblique derivative problem.` The oblique derivative problem was first formulated by Poincaré in his studies related to the theory of tides, and by the late 60's the two-dimensional setting was well-understood. At that time, much of the work in the multidimensional case has been restricted to the situation when the direction field of the derivatives is transversal to the boundary at each point, a condition which ensures that the ellipticity is nowhere violated. However, when the ellipticity degenerates, this problem turned out to be considerably more difficult and subtle. This case came under scrutiny in 60's when a series of papers were published in which the degenerate oblique derivative problem was considered in the scenario when the vector field is tangent to the boundary along a submanifold of codimension one, to which this vector is not tangent. This line of work received a big impetus when in 1970 Vladimir Maz'ya initiated a deep investigation of the problem in the case in which the boundary contains a nested family of submanifolds $\Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_s$ with the property that the vector field is tangent to Γ_k at points belonging to Γ_{k+1}, and is transversal to Γ_s. By employing a new technique, Vladimir Maz'ya was able to prove in this setting the unique solvability of the problem in a formulation which includes an additional Dirichlet condition on the entry set of the vector field and allows the possibility of discontinuities of the solution at points of the exit set. Up to now, this is the only known result pertaining to the oblique derivative problem in the generic situation in the sense of V. Arnold, who has considered this problem as an illustration of his calculus of infinite co-dimensions (see [**3**], §29 B). According to a hypothesis of Arnold, all submanifolds $\Gamma_1, \dots, \Gamma_s$, induce infinite dimensional kernels or co-kernels for the oblique derivative problem. Nonetheless, Maz'ya's striking theorem reveals that Arnold's hypothesis is inadequate, since it turns out that submanifolds of codimension greater than one in the boundary are negligible, in the sense that they play the same type of role as removable singularities.

◇ `Boundary-value problems in domains with piecewise-smooth boundaries.` Vladimir Maz'ya has started working in this field at the beginning of the 1960's and from his early publications he was able to establish deep and unexpected results regarding second-order elliptic equations. For example, in studying selfadjointness conditions for the Laplace operator with zero Dirichlet data on contours of class C^1 (but not C^2), he discovered a surprising instability effect for the index under affine coordinate transformations. Following the emergence of Kondrat'ev's well-known 1967 paper on elliptic boundary-value problems in domains with conic singularities, Vladimir Maz'ya began working actively in this field and, in collaboration with B.A. Plamenevskii, and later with V.A. Kozlov and J. Rossmann, has produced a string of papers which contain a fascinating theory of boundary-value problems in domains with piecewise smooth boundary, including regularity estimates, asymptotic representations of solutions, well-posedness theorems, and methods for computing the coefficients in the asymptotics of solutions near boundary singularities. The theory thus developed, together with important applications to problems arising in mechanics, engineering and mathematical physics, is presented in the monographs [Ma-7], [Ma-8], [Ma-15], and [Ma-16].

The aforementioned body of results complements the theory of elliptic boundary value problems in Lipschitz domains, as initiated by A. Calderón, B. Dahlberg, E. Fabes, N. Riviere, M. Jodeit, C. Kenig, D. Jerison, J. Pipher, G. Verchota starting in the late 70's and early 80's. An authoritative account of the state of the art in this field, up to the mid 90's can be found in C. Kenig's book [**7**]. Compared with the latter, the former setting of domains with piecewise smooth boundaries allows for a wide range of non-Lipschitz domains. A simple example is offered by Maz'ya's "two-brick domain":

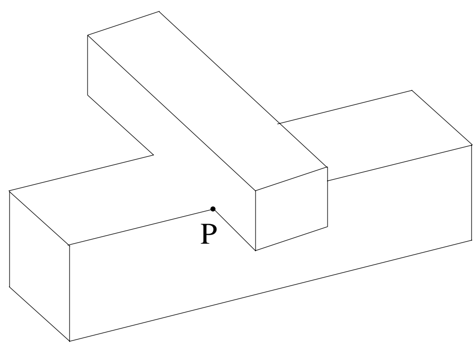

FIGURE 1

Indeed, a moment's reflection shows that near the point P, the boundary of the above domain is not the graph of any function (as it fails the vertical line test) even after applying a rigid motion. Most recently, progress in understanding such configurations from the Harmonic Analysis perspective has been recorded in [**12**], [**13**], [**14**].

◇ `Multipliers between spaces of differentiable functions.` In the late 70's, Vladimir Maz'ya and Tatyana Shaposhnikova initiated a systematic study of multipliers in pairs of various spaces of differentiable functions. This resulted in their joint book [Ma-19], which for the time being, is the only monograph on this topic. The forthcoming book [Ma-1] by the same authors reports on the more recent progress in this area. The obvious motivation for a thorough investigation of properties of multipliers stems from the study of partial differential equations of the type

$$Lu := \sum_{|\alpha|,|\beta|\leq m} \partial^\alpha (a_{\alpha,\beta}(X)\partial^\beta u) = f \quad \text{in} \quad \Omega, \tag{1}$$

in which the data and the solution belong to appropriate Sobolev spaces in the domain $\Omega \subset \mathbb{R}^n$. It is then of interest to understand how multiplication by the coefficients $a_{\alpha,\beta}$ transforms these classes of functions. A similar perspective comes from treating (1) via localization and flattening of the boundary of the domain Ω, for the purpose of transforming the original PDE into a problem in the upper half-space $\mathbb{R}^n_+$. In this scenario, the multiplier properties of the functions $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ which locally describe $\partial\Omega$ come into play. For example, in the case of the Poisson problem for the Laplacian with a Dirichlet boundary condition, i.e.,

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ \operatorname{Tr} u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

such techniques allow for a sharp description of the analytical properties of Ω required for the implication

$$f \in L^p(\Omega) \implies u \in W^{2,p}(\Omega) \tag{3}$$

to hold (where $1 < p < \infty$ is given). One other route through which multipliers take center-stage in a natural fashion is when one considers PDE's on manifolds, in which case the transformational properties of (1) under changes of variables are of central focus.

For general multipliers, Maz'ya and Shaposhnikova have established a wealth of basic results on the spectrum, traces and extensions, implicit functions, and two-sided estimates for the essential norm. They have also identified various classes of mappings and classes of non-smooth manifolds on which these multiplier spaces are invariantly defined. In addition, a calculus of singular integral operators with symbols in the space of multipliers was developed. These efforts have been amply rewarded by the fact that such a theory permits for deep applications to elliptic boundary value problems in domains with non-smooth boundaries.

◇ `Isoperimetric and integral inequalities, and theory of capacities.` While a fourth-year student at LSU, Vladimir Maz'ya made the remarkable discovery that integral inequalities of Sobolev type are actually equivalent to certain isoperimetric and isocapacitary inequalities for subsets of the domain where a function is defined. Even today, Vladimir likes to recall that special moment of inspiration, and he can artfully and fluidly reproduce the original calculations, to the delight of an interested interlocutor. These results, which eventually became part of his Ph.D. thesis, appeared in press in 1960-61. This original approach enabled him to obtain sharp constants in the aforementioned integral inequalities. In particular, the sharp constant in Gagliardo's inequality

$$\|u\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \le C_n \|\nabla u\|_{L^1(\mathbb{R}^n)}, \qquad u \in C_0^\infty(\mathbb{R}^n), \tag{4}$$

proved to be equal to that in the classical isoperimetric inequality: $C_n = n^{-1} v_n^{-1/n}$, where v_n is the volume of the unit ball in $\mathbb{R}^n$ (this was also found simultaneously and independently by G. Federer and W.H. Fleming). More importantly, as Maz'ya himself emphasized in 1966, his proofs did not make use of any specific properties of the Euclidean space and, hence, could be carried over to the setting of Riemannian manifolds.

An important inequality proved by Maz'ya (1964, 1972), and which later became known as the strong type capacitary inequality, allowed him to obtain capacitary criteria for Sobolev-type estimates. In more recent papers (2005, 2006), he has also obtained some important generalizations of this inequality. He also discovered (2003) that embeddings in fractional Besov spaces, or Riesz potential spaces, are equivalent with the validity of a certain new type of isoperimetric inequalities. The 1964, 1972 papers of Maz'ya, mentioned above, have motivated a thorough study of different aspects of the theory of Sobolev spaces, and have decisively influenced the development of this branch of mathematics. Currently, the methods in those papers are the driving force in the study of Sobolev spaces on metric spaces. The collection of results obtained up to 1985 are diligently presented in [Ma-20], arguably the most popular book authored by Vladimir Maz'ya (scheduled to appear in a new edition shortly).

The systematic use of the notion of capacity of a set eventually became a recurrent theme of a sizable number of Maz'ya's papers. As early as 1963 he introduced the polyharmonic capacity and successfully employed it in order to find optimal conditions for the well-posedness of the Dirichlet problem in the energy space for higher-order elliptic equations. At the beginning of 70's, V. Maz'ya and V.P. Khavin considered non-linear potentials and systematically studied their properties. Presently, the theory of non-linear potentials (naturally viewed as an extension of the classical linear theory) is a main-stream, active and fast-growing area of research, which has helped produce answers to many basic questions in the theory of functions, particularly for those concerning the nature of exceptional sets.

◇ `Theory of the Schrödinger operator.` By making essential use of his previously developed capacitary criteria, Vladimir Maz'ya was able to obtain (in 1962, 1964), sharp conditions ensuring the validity of various spectral properties of the Schrödinger operator. More recently, in their masterful 2002 Acta Mathematica paper, Vladimir G. Maz'ya and Igor E. Verbitsky have identified the correct class of complex-valued potentials for which the Schrödinger operator $-\Delta + V$ maps the energy space into its own dual. Subsequently, V. Maz'ya, V.A. Kondrat'ev and M.A. Shubin (2004) have proved necessary and sufficient conditions for the spectrum of the Schrödinger operator with a magnetic potential to be positive and discrete, thus generalizing the well-known work of A.M. Molchanov on this topic (who has treated the case when the magnetic field is absent). In 2005, V. Maz'ya and M. Shubin succeeded in characterizing the sets which are negligible in Molchanov's criterion, thereby solving an long-standing open problem, originally posed by I.M. Gel'fand in 1953.

◇ `Boundary behavior and maximum principles for elliptic and parabolic systems.` One of the prevalent themes of research throughout Vladimir Maz'ya's career, is the issue of regularity of a boundary point in the sense of Wiener. As early as 1962, he has proved an estimate for the modulus of continuity of a harmonic function, formulated in terms of the Wiener integral which, in turn, has found important applications in the qualitative theory of linear and non-linear elliptic equations. Then in 1970 he formulated a condition for regularity, in the sense of Wiener, of a boundary point for a certain class of quasi-linear second-order elliptic operators, which includes the p-Laplacian. Conspicuously, all these years virtually nothing was known about the Wiener type regularity of a boundary point for higher-order equations. The breakthrough came in 2002 when Vladimir Maz'ya succeeded in generalizing the Wiener test to elliptic equations of arbitrary order. Subsequently, this fundamental result made the subject of Maz'ya's talk at the International Congress of Mathematicians in Beijing.

In collaboration with G.I. Kresin, Vladimir Maz'ya has produced, in a series of papers starting around mid 80's, a necessary and sufficient condition formulated in algebraic terms guaranteeing the validity of the classical maximum modulus principle for second-order elliptic and parabolic systems. Next, in 1992, V. Maz'ya and J. Rossmann proved that the classical Miranda-Agmon maximum principle actually holds for any strongly elliptic operator of arbitrary order in a plane domain with a piece-wise smooth boundary, without peaks. While a similar result holds in the three-dimensional setting, in dimensions four and higher this principle fails for certain domains with conical vertices. For the polyharmonic (and biharmonic) equations in Lipschitz and C^1 domains, this issue has been further investigated in [**10**], [**11**].

◇ `Theory of water waves.` During his tenure at the Leningrad Shipbuilding Institute, Vladimir Maz'ya became interested in the mathematical theory of linear surface waves and, in 1973, wrote two articles in collaboration with B.R. Vainberg, in which the basic boundary value problems of this theory are studied. Four years later, Vladimir Maz'ya was the first to obtain a rather general uniqueness condition for the problem of oscillations of a body fully immersed in a liquid, which was originally stated by F. John as far back as 1950. The papers produced by Vladimir Maz'ya and his collaborators on this topic eventually led to the monograph [Ma-6].

Even from this brief review it is amply clear that Vladimir Maz'ya's work has an astonishing range and depth. However, he has left a lasting mark of originality and technical virtuosity in many more other branches of mathematics, such as *estimates for general differential and pseudodifferential operators in a half-space*, an area in which he has co-authored with I.V. Gel'man the monograph [Ma-21]; *Sobolev spaces and asymptotic theory of elliptic boundary-value problems on singularly perturbed domains*, in which Maz'ya has developed a rather sophisticated theory, first in collaboration with S.V. Poborchi, then jointly with S.A. Nazarov and B. Plamenevskii, as well as V.A. Kozlov and A.B. Movchan, which makes the subject of [Ma-8], [Ma-9], and [Ma-11], respectively, *numerical analysis* (cf. [Ma-2] written with G. Schmidt); *history of mathematics*, an area in which he has co-authored with Tatyana Shaposhnikova a delightful and highly informative book about the life and work of J. Hadamard ([Ma-5],[Ma-12]); *asymptotic theory of solutions to differential equations with operator coefficients* [Ma-10], written jointly with V.A. Kozlov; and *estimates for analytic functions with a bounded real part*, described in the book [Ma-3], based on the joint research with G. Kresin. This list should also include pointwise interpolation inequalities for derivatives, approximation by analytic and harmonic functions, degenerate elliptic pseudodifferential operators, uniqueness theorems for certain boundary value problems with data prescribed on only a portion of the boundary, characteristic Cauchy problems for hyperbolic equations, iterative procedures for solving ill-posed boundary value problems, etc.

Always animated by large, important ideas, magnanimous in sharing his expertise with other, particularly younger, people, one can only wonder what other magnificent contributions Vladimir Maz'ya will make in the future; we wish him many more years ahead, in good health.

V. Books (co-)authored by Vladimir Maz'ya.

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[Ma-2] *Approximate Approximations*, with G. Schmidt, American Mathematical Society, 2007.

[Ma-3] *Sharp Real-Part Theorems. A Unified approach*, with G. Kresin, Lecture Notes in Mathematics, No. 1903, Springer, 2007.

[Ma-4] *Imbedding and Extension Theorems for Functions in Non-Lipschitz Domains*, with S. Poborchi, St-Petersburg University Publishers, 2007.

[Ma-5] *Jacques Hadamard, un Mathématicien Universel*, with T. Shaposhnikova, EDP Sciences, Paris, 2005 (revised and extended translation from English).

[Ma-6] *Linear Water Waves. A Mathematical Approach*, with N. Kuznetsov and B. Vainberg, Cambridge University Press, 2002.

[Ma-7] *Spectral Problems Associated with Corner Singularities of Solutions to Elliptic Equations*, with V. Kozlov and J. Rossmann, Mathematical Surveys and Monographs, Vol. 85, American Mathematical Society, 2000.

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[Ma-11] *Asymptotic Analysis of Fields in Multistructures*, with V. Kozlov and A. Movchan, Oxford Science Publications, 1999

[Ma-12] *Jacques Hadamard, a Universal Mathematician*, with T. Shaposhnikova, American Mathematical Society and London Mathematical Society, 1998.

[Ma-13] *Differentiable Functions on Bad Domains*, with S. Poborchi, World Scientific, 1997.

[Ma-14] *Theory of a Higher-order Sturm-Liouville Equation*, with V. Kozlov, Springer-Verlag, Lecture Notes in Mathematics, 1997.

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[Ma-18] *Encyclopaedia of Mathematical Sciences, Vol. 26, Analysis III, Spaces of Differentiable Functions*, S.M. Nikol'skii (Ed.), Contributors: L.D. Kudryavtsev, V.G. Maz'ya, S.M. Nikol'skii, 218 pages, Springer-Verlag, 1990, V.G. Maz'ya: Classes of Domains, Measures and Capacities in the Theory of Differentiable Functions, pp. 141–211.

[Ma-19] *Theory of Multipliers in Spaces of Differentiable Functions*, with T. Shaposhnikova, Pitman, 1985 (Russian version: Leningrad University Press, 1986).

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