According to this definition, (4) does not converge. Good, but what about the following sequences:

(5) $$8,\ 1,\ 4,\ 1/2,\ 2,\ 1/4,\ 1,\ 1/8,\ 1/2,\ 1/16,\ 1/4,\ 1/32,\ \ldots$$

(6) $$1\frac{1}{2},\ 1\frac{1}{4},\ 1\frac{1}{8},\ 1\frac{1}{16},\ 1\frac{1}{32},\ \ldots$$

The terms of (5) are not getting "closer and closer to 0," but the sequence does converge to 0. The terms of (6) are getting "closer and closer to 0," but the sequence does not converge to 0. (We will see that this sequence converges to 1.)

We need a more precise definition. The terms need to get *and stay* arbitrarily close to zero or whatever limit value L, *eventually.* "Arbitrarily close" means as close as anyone could prescribe, i.e., given any positive error, the terms eventually have to stay within that tolerance of error. Since the letter e is already taken by 2.71828..., mathematicians usually use the Greek letter epsilon ε for the given error. And what do we mean by "eventually"? We mean that given the tolerance of error ε, we can come up with a big number N, such that all the terms after a_N are within ε of the limit value L. That is, given $\varepsilon > 0$, we can come up with an N, such that whenever $n > N$, every subsequent a_n is within ε of L. Now that's a good definition, and here it is written out concisely:

3.2. Definition of convergence. A sequence a_n *converges* to a *limit* L

$$a_n \to L$$

if given $\varepsilon > 0$, there is some N, such that whenever $n > N$,

$$|a_n - L| < \varepsilon.$$

Otherwise we say that the sequence *diverges.*

Notice that order in the definition is very important. First comes the sequence a_n and the proposed limit L. Second comes the tolerance of error ε, which is allowed to depend on the sequence. Third comes the N, which is allowed to depend on ε.

A sequence which diverges might diverge to infinity like (3) or diverge by oscillation like (2).

Some other things in the definition do not matter, such as whether the inequalities are strict or not. For example, if you knew only that you could get $|a_n - L|$ less than *or equal to* any given ε, you could take a new $\varepsilon' = \varepsilon/2$ and get

$$|a_n - L| \le \varepsilon' < \varepsilon,$$

strictly less than ε. Similarly, it suffices to get $|a_n - L|$ less than 3ε, because you can take $\varepsilon' = \varepsilon/3$ and get

$$|a_n - L| < 3\varepsilon' = \varepsilon.$$

So the final ε could be replaced by any constant times ε or anything that's small when ε is small.

3.3. Example of convergence. Prove that $a_n = 1/n^2$ converges to 0.

First, let's think it through. Given $\varepsilon > 0$, we have to see how big n has to be to guarantee that $a_n = 1/n^2$ is within ε of 0:

$$|1/n^2 - 0| < \varepsilon.$$

This will hold if $1/n^2 < \varepsilon$, that is, if $n > 1/\sqrt{\varepsilon}$. So we can just take N to be $1/\sqrt{\varepsilon}$, and we'll have the following proof:

$$\text{Given } \varepsilon > 0, \text{ let } N = 1/\sqrt{\varepsilon}. \text{ Then whenever } n > N,$$
$$|a_n - L| = |1/n^2 - 0| = 1/n^2 < 1/N^2 = \varepsilon.$$

Notice how we had to work backwards to come up with the proof.

3.4. Bounded. A sequence a_n is *bounded* if there is a number M such that for all n, $|a_n| \le M$.

For example, the sequence $a_n = \sin n$ is bounded by 1 (and by any $M \ge 1$). The sequence $a_n = (-1)^n/n^2$ is bounded by 1. The sequence $a_n = n^2$ is not bounded.

3.5. Proposition. *Suppose that the sequence a_n converges. Then*

(1) *the limit is unique;*

(2) *the sequence is bounded.*

Before starting the proof of (1), let's think about why a sequence cannot have two limits, 0 and 1/4 for example. It's easy for the terms a_n to get within 1 of both, or to get within 1/2 of both, but no better than within $\varepsilon = 1/8$ of both (see Figure 3.1).

Similarly, if a sequence had any two different limits $L_1 < L_2$, you should get a contradiction when $\varepsilon = (1/2)(L_2 - L_1)$. I think I'm ready to write the proof.

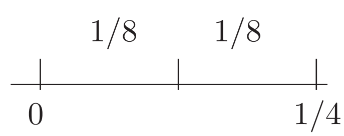


Figure 3.1. A number cannot be closer than distance 1/8 to both 0 and 1/4.

Proof of (1). Suppose that a sequence a_n converges to two different limits $L_1 < L_2$. Let $\varepsilon = (1/2)(L_2 - L_1)$. By the definition of convergence, there is some N_1 such that whenever $n > N_1$, $|a_n - L_1| < \varepsilon$. Similarly there is some N_2 such that whenever $n > N_2$, $|a_n - L_2| < \varepsilon$. Choose n greater than N_1 and N_2. Then

$$L_2 - L_1 \leq |a_n - L_1| + |L_2 - a_n| < \varepsilon + \varepsilon = 2\varepsilon = L_2 - L_1,$$

a contradiction. □

The proof of (2) is easier. After a while the sequence is close to its limit L, and once a_n is within say 1 of L, $|a_n| < |L| + 1$. There are only finitely many other terms to worry about, and of course any finite set is bounded (by its largest element). I'm ready to write the proof:

Proof of (2). Let a_n be a sequence converging to L. Choose N such that whenever $n > N$, $|a_n - L| < 1$, so that $|a_n| < |L| + 1$. Let

$$M = \max\{|L| + 1, |a_n|, \text{ with } n \leq N\}.$$

Then if $n \leq N$, $a_n \leq M$. If $n > N$, $a_n < |L| + 1 \leq M$. So always $a_n \leq M$. □

3.6. Proposition. *Suppose real sequences a_n, b_n converge to a and b:*

$$a_n \to a, \quad b_n \to b.$$

Then

(1) $ca_n \to ca$,

(2) $a_n + b_n \to a + b$,

(3) $a_n b_n \to ab$,

(4) $a_n/b_n \to a/b$, *assuming every b_n and b is a nonzero real number.*

Preparation for proof. We'll prove (1) and (4), and leave (2) and (3) as Exercises 13 and 14. As usual, it pays to build the proof backwards. At the end of the proof of (1), we'll need to estimate

$$|ca_n - ca| = |c|\,|a_n - a| < \varepsilon,$$

which will hold if $|a_n - a| < \varepsilon/|c|$ (unless $c = 0$). I see how to do the proof.

Proof. We may assume that $c \neq 0$, since that case is trivial (it just says that $0, 0, 0, \ldots \to 0$). Since c is a fixed constant, given $\varepsilon > 0$, since $a_n \to a$, we can choose N such that whenever $n > N$, $|a_n - a| < \varepsilon/|c|$. Then

$$|ca_n - ca| = |c|\,|a_n - a| < \varepsilon,$$

so that $ca_n \to ca$.

The proof of (4) is harder, so we start with a discussion. At the end of the proof, we'll need to estimate $|a_n/b_n - a/b|$ in terms of things we know are small: $|a_n - a|$ and $|b_n - b|$. The trick is an old one that you first see in the proof of the quotient rule in calculus: go from a_n/b_n to a/b in two steps, changing one part at a time, from a_n/b_n to a/b_n to a/b, to end up with some estimate like:

$$\begin{aligned} |a_n/b_n - a/b| &\leq |a_n/b_n - a/b_n| + |a/b_n - a/b| \\ &= |a_n - a|/|b_n| + |b - b_n|\,|a/bb_n|. \end{aligned}$$

We know that $|a_n - a|$ and $|b - b_n|$ are small, and $|a/b|$ is just a constant, but what about those $1/|b_n|$? We need to know that b_n is not too close to 0. Fortunately since $b_n \to b$, eventually $|b_n| > |b|/2$ (as soon as b_n gets within $|b|/2$ of b). Then $1/|b_n| \leq 2/|b|$. So the estimate can continue

$$\begin{aligned} &\leq |a_n - a|(2/|b|) + |b - b_n|\,|2a/b^2| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon, \end{aligned}$$

if we just make sure that $|a_n - a|(2/|b|)$ and $|b - b_n|\,|2a/b^2|$ are less than $\varepsilon/2$ by making $|a_n - a| < \varepsilon|b|/4$ and $|b - b_n| < \varepsilon|b^2/4a|$ (which we interpret as no condition on b_n if $a = 0$). Can we guarantee both of those conditions at the same time? We can find an N_1 to make the first one hold for $n > N_1$, and we can find an N_2 to make the second one hold for $n > N_2$. To make both work, just take N to be the maximum of N_1 and N_2. In general, you can always handle finitely many conditions.

Here's the whole proof from start to finish. Since $a_n \to a$ and $b_n \to b$, we can choose N such that whenever $n > N$, the following hold:

$$\begin{aligned} &|a_n - a| < \varepsilon|b|/4, \\ &|b - b_n| < \varepsilon|b^2/4a|, \quad \text{and} \\ &|b - b_n| < |b|/2, \quad \text{which implies that } 1/|b_n| \leq 2/|b|. \end{aligned}$$

Then

$$\begin{aligned} |a_n/b_n - a/b| &\leq |a_n/b_n - a/b_n| + |a/b_n - a/b| \\ &= |a_n - a|/|b_n| + |b - b_n|\,|a/bb_n| \\ &\leq |a_n - a|(2/|b|) + |b - b_n|\,|2a/b^2| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon, \end{aligned}$$

and consequently $a_n/b_n \to a/b$. □

It would have been OK and simpler to start out by just requiring that

$$\begin{aligned} &|a_n - a| < \varepsilon, \\ &|b - b_n| < \varepsilon, \quad \text{and} \\ &|b - b_n| < |b|/2, \quad \text{which implies that} \quad 1/|b_n| \leq 2/|b|. \end{aligned}$$

Then

$$\begin{aligned}|a_n/b_n - a/b| &\le |a_n/b_n - a/b_n| + |a/b_n - a/b| \\ &= |a_n - a|/|b_n| + |b - b_n|\,|a/bb_n| \\ &< \varepsilon(2/|b|) + \varepsilon|2a/b^2| = C\varepsilon,\end{aligned}$$

where C is the constant $(2/|b|)+|2a/b^2|$. Although we haven't made it come out quite as neatly at the end, we've still shown that we can make the error arbitrarily small by choosing n large enough, which is sufficient.

3.7. Rates of growth. Limits of many sequences can be determined just by knowing that for n large,

$$1/n^2 \ll 1/n \ll 1 \ll \ln n \ll \sqrt{n} \ll n \ll n^2 \ll n^3 \ll 2^n \ll e^n \ll 10^n \ll n!$$

where $f(n) \ll g(n)$ means that f becomes a negligible percentage of g, $f(n)/g(n) \to 0$, so that in a limit as $n \to \infty$ whenever you see $f+g$, or even $c_1 f + c_2 g$, you can ignore the negligible f. For example,

$$\lim_{n\to\infty} \frac{n^4 + \ln(n+1)}{\sqrt{5n^8+16}} = \lim_{n\to\infty} \frac{n^4}{\sqrt{5n^8}} = \frac{1}{\sqrt{5}}.$$

3.8. Three famous limits.

(1) $$\sqrt[n]{2} = 2^{1/n} \to 2^0 = 1.$$

(2) $$\sqrt[n]{n} = n^{1/n} = (e^{\ln n})^{1/n} = e^{(\ln n)/n} \to e^0 = 1.$$

(The exponent $(\ln n)/n \to 0$ because $\ln n \ll n$.)

(3) $$(1+1/n)^n \to e.$$

(This is sometimes used as the definition of the number e, after you check that the limit exists. Exercise 25.6 derives it from another definition.)

3.9. Accumulation points. A point p is an *accumulation point* of a set S if it is the limit of a sequence of points of $S - \{p\}$. It is equivalent to require that every ball $(p-r, p+r)$ about p intersect $S - \{p\}$ (Exercise 19).

For example, 0 is an accumulation point of $\{1/n \colon n \in \mathbb{N}\}$ because $0 = \lim 1/n$ or because every ball (interval) about 0 intersects $\{1/n\}$. In this case the accumulation point is not in the original set.

Every point of the unit interval $[0,1]$ is an accumulation point. In this case all of the accumulation points are in the set.

3.10. $\mathbb{R}^n$. Almost everything in this chapter works for vectors in $\mathbb{R}^n$ as well as for points in $\mathbb{R}$. The exception is Proposition 3.6(4) because there is no way to divide vectors. Proposition 3.6(3) holds both for the dot product and for the cross product.

Exercises 3

Does the sequence converge or diverge? If it converges, what is the limit?

1. 1, 0, 1/2, 0, 1/4, 0, 1/8, 0, ...

2. 3, 3.1, 3.14, 3.141, 3.1415, 3.14159, ...

3. $a_n = 1 + (-1)^n/n$.

4. $a_n = \frac{1+(-1)^n}{n}$.

5. $a_n = (-1)^n(1 - 1/n)$.

6. $a_n = 1 + (-1)^n$.

7. $a_n = \frac{2n^2+5n+1}{7n^2+4n+3}$.

8. $a_n = \frac{e^n}{n^5+n-5}$.

9. $a_n = \frac{2^n}{n!}$.

10. $a_n = \frac{\sin n}{n}$.

11. Prove that $a_n = 1/n$ converges to 0.

12. Prove that $a_n = 1000/n^3$ converges to 0.

13. Prove 3.6(2).

14. Prove 3.6(3).

15. Prove that if $a_n \le b_n \le c_n$ and $\lim a_n = \lim c_n = L$ then $\lim b_n = L$.

16. Prove or give a counterexample. Let a_n be a sequence such that $a_{n+1} - a_n \to 0$. Does a_n have to converge?

17. A sequence a_n is called *Cauchy* if, given $\varepsilon > 0$, there is an N such that whenever $m, n > N$, $|a_m - a_n| < \varepsilon$. Prove that if a sequence in $\mathbb{R}$ is convergent, then it is Cauchy. (Exercise 8.6 will prove the converse in $\mathbb{R}$, so that Cauchy gives a nice criterion for convergence without mentioning what the limit is.)

18. Prove that a Cauchy sequence is bounded.

19. Describe the set of accumulation points of the following sets:

a. the rationals;

b. the irrationals;

c. $[a, b)$;

d. the integers.

20. Prove that p is an accumulation point of S if and only if every ball B about p intersects $S - \{p\}$.

21. Prove or give a counterexample: There are only countably many sequences with limit 0.

22. Prove or give a counterexample: a real increasing sequence

$$a_1 < a_2 < a_3 < \cdots$$

converges if and only if the differences $a_{n+1} - a_n$ converge to 0.

Chapter 9

Compactness

Probably the most important new idea you'll encounter in real analysis is the concept of compactness. It's the compactness of $[a, b]$ that makes a continuous function reach its maximum and that makes the Riemann integral exist. For subsets of $\mathbb{R}^n$, there are three equivalent definitions of compactness. The first, 9.2(1), promises convergent subsequences. The second, 9.2(2) brings together two apparently unrelated adjectives, *closed* and *bounded*. The third, 9.2(3), is the elegant, modern definition in terms of open sets; it is very powerful, but it takes a while to get used to.

9.1. Definitions. Let S be a set in $\mathbb{R}^n$. S is *bounded* if it is contained in some ball $B(0, R)$ about 0 (or equivalently in a ball about any point). A collection of open sets $\{U_\alpha\}$ is an *open cover* of S if S is contained in $\bigcup U_\alpha$. A *finite subcover* is finitely many of the U_α which still cover S. Following Heine and Borel, S is *compact* if every open cover has a finite subcover.

9.2. Theorem. Compactness. *The following are all equivalent conditions on a set S in $\mathbb{R}^n$.*

(1) *Every sequence in S has a subsequence converging to a point of S.*

(2) *S is closed and bounded.*

(3) *S is compact: every open cover has a finite subcover.*

Criterion (1) is the Bolzano–Weierstrass condition for compactness, which you met for $\mathbb{R}$ in Theorem 8.3. The more modern Heine–Borel criterion (3) will take some time to get used to. A nonclosed set such as $(0, 1]$ is not compact because the open cover $\{(1/n, \infty)\}$ has no finite subcover. An unbounded set such as $\mathbb{R}$ is not compact because the open cover $\{(-n, n)\}$ has no finite subcover. This is the main idea of the first part of the proof.

Proof. We will prove that (3) ⇒ (2) ⇒ (1) ⇒ (3).

(3) ⇒ (2). Suppose that S is not closed. Let a be an accumulation point not in S. Then the open cover $\{\{|x-a| > 1/n\}\}$ has no finite subcover. Suppose that S is not bounded. Then the open cover $\{\{|x| < n\}\}$ has no finite subcover.

(2) ⇒ (1). Take any sequence of points in $S \subset \mathbb{R}^n$. First look at just the first of the n components of each point. Since S is bounded, the sequence of first components is bounded. By Theorem 8.3, for some subsequence, the first components converge. Similarly, for some further subsequence, the second components also converge. Eventually, for some subsequence, all of the components converge. Since S is closed, the limit is in S.

(1) ⇒ (3). Given an open cover $\{U_\alpha\}$, first we find a *countable* subcover. Indeed, every point x of S lies in a ball of rational radius about a rational point, contained in some U_α. Each of these countably many balls lies in some U_α. Let $\{V_i\}$ be that countable subcover.

Suppose that $\{V_i\}$ has no finite subcover. Choose x_1 in S but not in V_1. Choose x_2 in S but not in $V_1 \cup V_2$. Continue, choosing x_n in S but not in $\bigcup\{V_i\colon 1 \leq i \leq n\}$, which is always possible because there is no finite subcover. Note that for each i, only finitely many x_n (for which $n < i$) lie in V_i. By (1), the sequence x_n has a subsequence converging to some x in S, contained in some V_i. Hence infinitely x_n are contained in V_i, a contradiction. □

9.3. Proposition. *A nonempty compact set S of real numbers has a largest element* (*called the* maximum) *and a smallest element* (*called the* minimum).

Proof. We may assume that S has some positive numbers, by translating it to the right if necessary. Since S is bounded, there is a largest integer part D before the decimal place. Among the elements of S that start with D, there is a largest first decimal place d_1. Among the elements of S that start with $D.d_1$, there is a largest second decimal place d_2. Keep going to construct $a = D.d_1d_2d_3\ldots.$ By construction, a is in the closure of S. Since S is closed, a lies in S and provides the desired maximum.

A minimum is provided by $-\max(-S)$. □

Exercises 9

1. Prove that the intersection of two compact sets is compact, using criterion (2).

2. Prove that the intersection of two compact sets is compact, using criterion (1).

3. Prove that the intersection of two compact sets is compact, using criterion (3).

4. Prove that the intersection of infinitely many compact sets is compact.

5. Prove that the union of two compact sets is compact, using criterion (2).

6. Prove that the union of two compact sets is compact, using criterion (1).

7. Prove that the union of two compact sets is compact, using criterion (3).

8. Is the union of infinitely many compact sets always compact? Give a proof or counterexample.

9. Identify the class of sets in $\mathbb{R}$ characterized by the following conditions and give two examples of such sets.

a. Every open cover has a finite subcover.

b. Every open cover has a countable subcover.

c. Some open cover has a finite subcover.

d. Every closed cover has a finite subcover.

10. Prove directly that (1) implies (2).

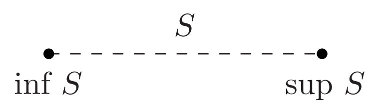
11. (Kristin Bohnhorst). Prove directly that (3) implies (1).

Hint: Assume that some (infinite) sequence x_n has no convergent subsequence. Then every point p of S has an open ball U_p around it which contains x_n for only finitely many n.

12. Is the converse of Proposition 9.3 true? Give a proof or counterexample.

13. Prove that if a nonempty closed set S of real numbers is bounded above, then it has a largest element.

14. Define the *supremum* $\sup S$ of a nonempty bounded set of real numbers as $\max \bar{S}$. Prove that $\sup S \geq s$ for all s in S and that $\sup S$ is the smallest number with that property. For this reason, $\sup S$ is often called the *least upper bound*. Similarly, the *infimum* $\inf S = \min \bar{S}$ is called the *greatest lower bound*.



(If S is not bounded above, $\sup S$ is defined to be $+\infty$. If S is empty, $\sup S$ is defined to be $-\infty$.)

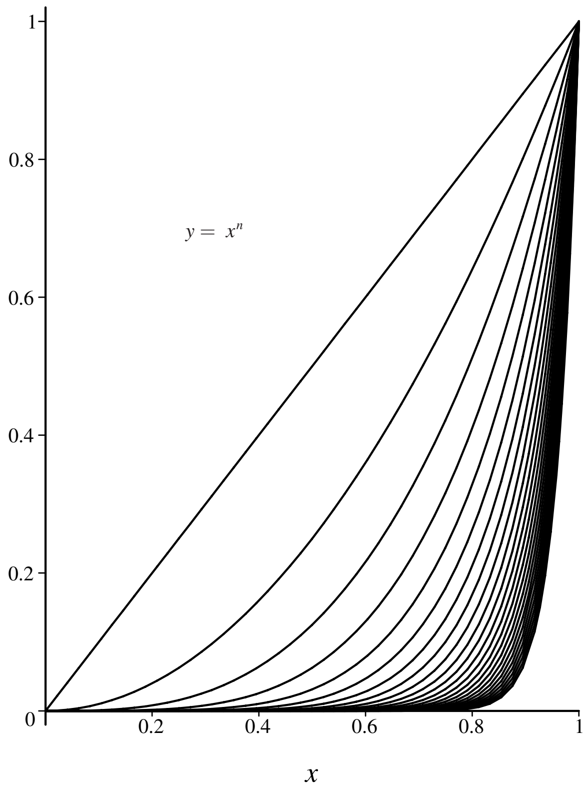


Figure 17.1. The continuous functions $f_n(x) = x^n$ converge pointwise to a discontinuous function.

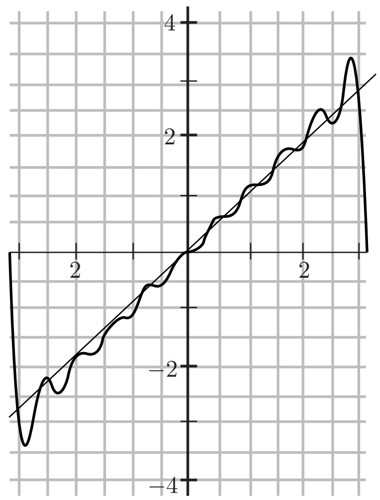


Figure 24.1. The Fourier approximations to $f(x) = x$ overshoot by about 9% at the endpoints, the so-called Gibbs phenomenon.