Preface to the Series

> Young men should prove theorems, old men should write books.
>
> —*Freeman Dyson*, quoting G. H. Hardy[1]

Reed–Simon[2] starts with "Mathematics has its roots in numerology, geometry, and physics." This puts into context the division of mathematics into algebra, geometry/topology, and analysis. There are, of course, other areas of mathematics, and a division between parts of mathematics can be artificial. But almost universally, we require our graduate students to take courses in these three areas.

This five-volume series began and, to some extent, remains a set of texts for a basic graduate analysis course. In part it reflects Caltech's three-terms-per-year schedule and the actual courses I've taught in the past. Much of the contents of Parts 1 and 2 (Part 2 is in two volumes, Part 2A and Part 2B) are common to virtually all such courses: point set topology, measure spaces, Hilbert and Banach spaces, distribution theory, and the Fourier transform, complex analysis including the Riemann mapping and Hadamard product theorems. Parts 3 and 4 are made up of material that you'll find in some, but not all, courses—on the one hand, Part 3 on maximal functions and H^p-spaces; on the other hand, Part 4 on the spectral theorem for bounded self-adjoint operators on a Hilbert space and det and trace, again for Hilbert space operators. Parts 3 and 4 reflect the two halves of the third term of Caltech's course.

[1] Interview with D. J. Albers, The College Mathematics Journal, **25**, no. 1, January 1994.

[2] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, I: Functional Analysis*, Academic Press, New York, 1972.

While there is, of course, overlap between these books and other texts, there are some places where we differ, at least from many:

(a) By having a unified approach to both real and complex analysis, we are able to use notions like contour integrals as Stietljes integrals that cross the barrier.

(b) We include some topics that are not standard, although I am surprised they are not. For example, while discussing maximal functions, I present Garcia's proof of the maximal (and so, Birkhoff) ergodic theorem.

(c) These books are written to be keepers—the idea is that, for many students, this may be the last analysis course they take, so I've tried to write in a way that these books will be useful as a reference. For this reason, I've included "bonus" chapters and sections—material that I do not expect to be included in the course. This has several advantages. First, in a slightly longer course, the instructor has an option of extra topics to include. Second, there is some flexibility—for an instructor who can't imagine a complex analysis course without a proof of the prime number theorem, it is possible to replace all or part of the (nonbonus) chapter on elliptic functions with the last four sections of the bonus chapter on analytic number theory. Third, it is certainly possible to take all the material in, say, Part 2, to turn it into a two-term course. Most importantly, the bonus material is there for the reader to peruse long after the formal course is over.

(d) I have long collected "best" proofs and over the years learned a number of ones that are not the standard textbook proofs. In this regard, modern technology has been a boon. Thanks to Google books and the Caltech library, I've been able to discover some proofs that I hadn't learned before. Examples of things that I'm especially fond of are Bernstein polynomials to get the classical Weierstrass approximation theorem, von Neumann's proof of the Lebesgue decomposition and Radon–Nikodym theorems, the Hermite expansion treatment of Fourier transform, Landau's proof of the Hadamard factorization theorem, Wielandt's theorem on the functional equation for $\Gamma(z)$, and Newman's proof of the prime number theorem. Each of these appears in at least some monographs, but they are not nearly as widespread as they deserve to be.

(e) I've tried to distinguish between central results and interesting asides and to indicate when an interesting aside is going to come up again later. In particular, all chapters, except those on preliminaries, have a listing of "Big Notions and Theorems" at their start. I wish that this attempt to differentiate between the essential and the less essential didn't make this book different, but alas, too many texts are monotone listings of theorems and proofs.

(f) I've included copious "Notes and Historical Remarks" at the end of each section. These notes illuminate and extend, and they (and the Problems) allow us to cover more material than would otherwise be possible. The history is there to enliven the discussion and to emphasize to students that mathematicians are real people and that "may you live in interesting times" is truly a curse. Any discussion of the history of real analysis is depressing because of the number of lives ended by the Nazis. Any discussion of nineteenth-century mathematics makes one appreciate medical progress, contemplating Abel, Riemann, and Stieltjes. I feel knowing that Picard was Hermite's son-in-law spices up the study of his theorem.

On the subject of history, there are three cautions. First, I am not a professional historian and almost none of the history discussed here is based on original sources. I have relied at times—horrors!—on information on the Internet. I have tried for accuracy but I'm sure there are errors, some that would make a real historian wince.

A second caution concerns looking at the history assuming the mathematics we now know. Especially when concepts are new, they may be poorly understood or viewed from a perspective quite different from the one here. Looking at the wonderful history of nineteenth-century complex analysis by Bottazzini–Grey[3] will illustrate this more clearly than these brief notes can.

The third caution concerns naming theorems. Here, the reader needs to bear in mind Arnol'd's principle:[4] *If a notion bears a personal name, then that name is not the name of the discoverer* (and the related Berry principle: *The Arnol'd principle is applicable to itself*). To see the applicability of Berry's principle, I note that in the wider world, Arnol'd's principle is called "Stigler's law of eponymy." Stigler[5] named this in 1980, pointing out it was really discovered by Merton. In 1972, Kennedy[6] named Boyer's law *Mathematical formulas and theorems are usually not named after their original discoverers* after Boyer's book.[7] Already in 1956, Newman[8] quoted the early twentieth-century philosopher and logician A. N. Whitehead as saying: "Everything of importance has been said before by somebody who

[3]U. Bottazzini and J. Gray, *Hidden Harmony—Geometric Fantasies. The Rise of Complex Function Theory*, Springer, New York, 2013.

[4]V. I. Arnol'd, *On teaching mathematics*, available online at `http://pauli.uni-muenster.de/~munsteg/arnold.html`.

[5]S. M. Stigler, *Stigler's law of eponymy*, Trans. New York Acad. Sci. **39** (1980), 147–158.

[6]H. C. Kennedy, *Classroom notes: Who discovered Boyer's law?*, Amer. Math. Monthly **79** (1972), 66–67.

[7]C. B. Boyer, *A History of Mathematics*, Wiley, New York, 1968.

[8]J. R. Newman, *The World of Mathematics*, Simon & Schuster, New York, 1956.

did not discover it." The main reason to give a name to a theorem is to have a convenient way to refer to that theorem. I usually try to follow common usage (even when I know Arnol'd's principle applies).

I have resisted the temptation of some text writers to rename things to set the record straight. For example, there is a small group who have attempted to replace "WKB approximation" by "Liouville–Green approximation", with valid historical justification (see the Notes to Section 15.5 of Part 2B). But if I gave a talk and said I was about to use the Liouville–Green approximation, I'd get blank stares from many who would instantly know what I meant by the WKB approximation. And, of course, those who try to change the name also know what WKB is! Names are mainly for shorthand, not history.

These books have a wide variety of problems, in line with a multiplicity of uses. The serious reader should at least skim them since there is often interesting supplementary material covered there.

Similarly, these books have a much larger bibliography than is standard, partly because of the historical references (many of which are available online and a pleasure to read) and partly because the Notes introduce lots of peripheral topics and places for further reading. But the reader shouldn't consider for a moment that these are intended to be comprehensive—that would be impossible in a subject as broad as that considered in these volumes.

These books differ from many modern texts by focusing a little more on special functions than is standard. In much of the nineteenth century, the theory of special functions was considered a central pillar of analysis. They are now out of favor—too much so—although one can see some signs of the pendulum swinging back. They are still mainly peripheral but appear often in Part 2 and a few times in Parts 1, 3, and 4.

These books are intended for a second course in analysis, but in most places, it is really previous exposure being helpful rather than required. Beyond the basic calculus, the one topic that the reader is expected to have seen is metric space theory and the construction of the reals as completion of the rationals (or by some other means, such as Dedekind cuts).

Initially, I picked "A Course in Analysis" as the title for this series as an homage to Goursat's *Cours d'Analyse*,[9] a classic text (also translated into English) of the early twentieth century (a literal translation would be

[9]E. Goursat, *A Course in Mathematical Analysis: Vol. 1: Derivatives and Differentials, Definite Integrals, Expansion in Series, Applications to Geometry. Vol. 2, Part 1: Functions of a Complex Variable. Vol. 2, Part 2: Differential Equations. Vol. 3, Part 1: Variation of Solutions. Partial Differential Equations of the Second Order. Vol. 3, Part 2: Integral Equations. Calculus of Variations*, Dover Publications, New York, 1959 and 1964; French original, 1905.

"of Analysis" but "in" sounds better). As I studied the history, I learned that this was a standard French title, especially associated with École Polytechnique. There are nineteenth-century versions by Cauchy and Jordan and twentieth-century versions by de la Vallée Poussin and Choquet. So this is a well-used title. The publisher suggested adding "Comprehensive", which seems appropriate.

It is a pleasure to thank many people who helped improve these texts. About 80% was TEXed by my superb secretary of almost 25 years, Cherie Galvez. Cherie was an extraordinary person—the secret weapon to my productivity. Not only was she technically strong and able to keep my tasks organized but also her people skills made coping with bureaucracy of all kinds easier. She managed to wind up a confidant and counselor for many of Caltech's mathematics students. Unfortunately, in May 2012, she was diagnosed with lung cancer, which she and chemotherapy valiantly fought. In July 2013, she passed away. I am dedicating these books to her memory.

During the second half of the preparation of this series of books, we also lost Arthur Wightman and Ed Nelson. Arthur was my advisor and was responsible for the topic of my first major paper—perturbation theory for the anharmonic oscillator. Ed had an enormous influence on me, both via the techniques I use and in how I approach being a mathematician. In particular, he taught me all about closed quadratic forms, motivating the methodology of my thesis. I am also dedicating these works to their memory.

After Cherie entered hospice, Sergei Gel'fand, the AMS publisher, helped me find Alice Peters to complete the TEXing of the manuscript. Her experience in mathematical publishing (she is the "A" of A K Peters Publishing) meant she did much more, for which I am grateful.

This set of books has about 150 figures which I think considerably add to their usefulness. About half were produced by Mamikon Mnatsakanian, a talented astrophysicist and wizard with Adobe Illustrator. The other half, mainly function plots, were produced by my former Ph.D. student and teacher extraordinaire Mihai Stoiciu (used with permission) using Mathematica. There are a few additional figures from Wikipedia (mainly under WikiCommons license) and a hyperbolic tiling of Douglas Dunham, used with permission. I appreciate the help I got with these figures.

Over the five-year period that I wrote this book and, in particular, during its beta-testing as a text in over a half-dozen institutions, I received feedback and corrections from many people. In particular, I should like to thank (with apologies to those who were inadvertently left off): Tom Alberts, Michael Barany, Jacob Christiansen, Percy Deift, Tal Einav, German Enciso, Alexander Eremenko, Rupert Frank, Fritz Gesztesy, Jeremy Gray, Leonard Gross, Chris Heil, Mourad Ismail, Svetlana Jitomirskaya, Bill Johnson, Rowan Killip, John Klauder, Seung Yeop Lee, Milivoje Lukic, Andre Martinez-Finkelshtein, Chris Marx, Alex Poltoratski, Eric Rains, Lorenzo Sadun, Ed Saff, Misha Sodin, Dan Stroock, Benji Weiss, Valentin Zagrebnov, and Maxim Zinchenko.

Much of these books was written at the tables of the Hebrew University Mathematics Library. I'd like to thank Yoram Last for his invitation and Naavah Levin for the hospitality of the library and for her invaluable help.

This series has a Facebook page. I welcome feedback, questions, and comments. The page is at `www.facebook.com/simon.analysis` .

Even if these books have later editions, I will try to keep theorem and equation numbers constant in case readers use them in their papers.

Finally, analysis is a wonderful and beautiful subject. I hope the reader has as much fun using these books as I had writing them.

Preface to Part 1

> I warn you in advance that all the principles ... that I'll now tell you about, are a little false. Counterexamples can be found to each one—but as directional guides the principles still serve a useful purpose.
>
> —*Paul Halmos*[1]

Analysis is the infinitesimal calculus writ large. Calculus as taught to most high school students and college freshmen is the subject as it existed about 1750—I've no doubt that Euler could have gotten a perfect score on the Calculus BC advanced placement exam. Even "rigorous" calculus courses that talk about ε-δ proofs and the intermediate value theorem only bring the subject up to about 1890 after the impact of Cauchy and Weierstrass on real variable calculus was felt.

This volume can be thought of as the infinitesimal calculus of the twentieth century. From that point of view, the key chapters are Chapter 4, which covers measure theory—the consummate integral calculus—and the first part of Chapter 6 on distribution theory—the ultimate differential calculus.

But from another point of view, this volume is about the triumph of abstraction. Abstraction is such a central part of modern mathematics that one forgets that it wasn't until Fréchet's 1906 thesis that sets of points with no a priori underlying structure (not assumed points in or functions on $\mathbb{R}^n$) are considered and given a structure a posteriori (Fréchet first defined abstract metric spaces). And after its success in analysis, abstraction took over significant parts of algebra, geometry, topology, and logic.

[1] L. Gillman, P. R. Halmos, H. Flanders, and B. Shube, *Four Panel Talks on Publishing*, Amer. Math. Monthly **82** (1975), 13–21.

Abstract spaces are a distinct thread here, starting with topological spaces in Chapter 2, Banach spaces in Chapter 5 (and its special case, Hilbert spaces, in Chapter 3), and locally convex spaces in the later parts of Chapters 5 and 6 and in Chapter 9.

Of course, abstract spaces occur to set up the language we need for measure theory (which we do initially on compact Hausdorff spaces and where we use Banach lattices as a tool) and for distributions which are defined as duals of some locally convex spaces.

Besides the main threads of measure theory, distributions, and abstract spaces, several leitmotifs can be seen: Fourier analysis (Sections 3.5, 6.2, and 6.4–6.6 are a minicourse), probability (Bonus Chapter 7 has the basics, but it is implicit in much of the basic measure theory), convexity (a key notion in Chapter 5), and at least bits and pieces of the theory of ordinary and partial differential equations.

The role of pivotal figures in real analysis is somewhat different from complex analysis, where three figures—Cauchy, Riemann, and Weierstrass—dominated not only in introducing the key concepts, but many of the most significant theorems. Of course, Lebesgue and Schwartz invented measure theory and distributions, respectively, but after ten years, Lebesgue moved on mainly to pedagogy and Hörmander did much more to cement the theory of distributions than Schwartz. On the abstract side, F. Riesz was a key figure for the 30 years following 1906, with important results well into his fifties, but he doesn't rise to the dominance of the complex analytic three.

In understanding one part of the rather distinct tone of some of this volume, the reader needs to bear in mind "Simon's three kvetches":[2]

1. Every interesting topological space is a metric space.
2. Every interesting Banach space is separable.
3. Every interesting real-valued function is Baire/Borel measurable.

Of course, the principles are well-described by the Halmos quote at the start—they aren't completely true but capture important ideas for the reader to bear in mind. As a mathematician, I cringe at using the phrase "not completely true." I was in a seminar whose audience included Ed Nelson, one of my teachers. When the speaker said the proof he was giving was almost rigorous, Ed said: "To say something is almost rigorous makes as much sense as saying a woman is almost pregnant." On the other hand, Neils Bohr, the founding father of quantum mechanics, said: "It is the hallmark of any deep truth that its negation is also a deep truth."[3]

[2] `http://www.merriam-webster.com/dictionary/kvetch`

[3] Quoted by Max Delbruck, *Mind from Matter? An Essay on Evolutionary Epistemology*, Blackwell Scientific Publications, Palo Alto, CA, 1986; page 167.

We'll see that weak topologies on infinite-dimensional Banach spaces are never metrizable (see Theorem 5.7.2) nor is the natural topology on $C_0^\infty(\mathbb{R}^\nu)$ (see Theorem 9.1.5), so Kvetch 1 has counterexamples, but neither case is so far from metrizable: If X^* is separable, the weak topology restricted to the unit ball of X is metrizable (see Theorem 5.7.2). While $C_0^\infty(\mathbb{R}^\nu)$ is not metrizable, that is because we allow ordinary distributions of arbitrary growth. If we restrict ourselves to distributions of any growth restriction, the test function space will be metrizable (see Sections 6.1 and 6.2). But the real point of Kvetch 1 is that the reason for studying topological spaces is *not* (merely) to be able to discuss nonmetrizable spaces—it is because metrics have more structure than is needed—$(0,1)$ is not complete with its usual metric while $\mathbb{R}$ is, but they are the same as topological spaces. Topological spaces provide the proper language for parts of analysis.

$L^\infty([0,1],dx)$ and $\mathcal{L}(\mathcal{H})$, the bounded operators on a Hilbert space, $\mathcal{H}$, are two very interesting spaces which are *not* separable, so Kvetch 2 isn't strictly true. But again, there is a point to Kvetch 2. In many cases, the most important members of a class of spaces are separable and one has to do considerable gymnastics in the general case, which is never, or at most very rarely, used. Of course, the gymnastics can be fun, but they don't belong in a first course. We illustrate this by including separability as an axiom for Hilbert spaces. Von Neumann did also in his initial work, but over the years, this has been dropped in most books. We choose to avoid the complications and mainly restrict ourselves to the separable case.

Two caveats: First, the consideration of the nonseparable case can provide more elegant proofs! For example, the projection lemma of Theorem 3.2.3 was proven initially for the separable case using a variant of Gram–Schmidt. The elegant proof we use that exploits convex minimization was only discovered because of a need to handle the nonseparable case. Second, we abuse the English language. A "red book" is a "book." We include separability and complex field in our definition of Hilbert space. We'll use the terms "nonseparable Hilbert space" and "real Hilbert space," which are not Hilbert spaces!

In one sense, Kvetch 3 isn't true, but except for one caveat, it is. Every set, A, has its characteristic function associated with it. If the only interesting functions are Borel functions, the only interesting sets are Borel sets. While it is a more advanced topic that we won't consider, there are sets constructed from Borel sets, called analytic sets and Souslin sets which may not be Borel.[4] The kvetch is there to eliminate Lebesgue measurable sets and functions, that is, sets $A = B \triangle C$, where B is Borel, and $C \subset D$, a Borel set of Lebesgue measure zero. The end of Section 4.3 discusses why it is not

[4]See, e.g., V. Bogachev, *Measure Theory*, Springer, 2007.

a good idea to consider such sets (and functions) even though many books do and it's what the Carathéodory construction of Section 8.1 leads to.

The last issue we mention in this preface is that our approach to measure theory is different from the standard one—it follows an approach in the appendix of Lax[5] that starts with a positive functional, ℓ, on $C(X)$, completes $C(X)$ in the $\ell(|f|)$-norm, and shows that the elements of the completion are equivalence classes of Borel functions. For those who prefer more traditional approaches, Section 4.13 discusses general measure spaces and Section 8.1 discusses the Carathéodory outer measure construction.

[5]P. Lax, *Functional Analysis*, Wiley, 2002.