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# Dynamical Properties of Diffeomorphisms of the Annulus and of the Torus

Patrice Le Calvez

*Translated by*  
Philippe Mazaud



American Mathematical Society  
Société Mathématique de France

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## Preface

Twist diffeomorphisms of the annulus can be found in a number of dynamical systems, both conservative and dissipative. An important research problem is that of finding periodic orbits, or orbits that have specific dynamical properties. Various methods have been devised for that purpose; some are variational (Aubry-Mather Theory) and are suited to the conservative case, others are topological and also apply to the dissipative case. In the first chapter, we shall give an overview of these methods. In the second chapter, we shall show that certain ideas from the Aubry-Mather Theory generalize in a natural way to the composition of twist maps, even in the dissipative case, enabling us to recover and even sharpen certain results on diffeomorphisms of the annulus or the two-torus that do not have the twist property.

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